

J. von Neumann's views on mathematical and axiomatic physics

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“Soft axiomatism: John von Neumann on method and von Neumann’s method in the physical sciences”, *Forthcoming in Intuition and the Axiomatic Method* eds.:

Emily Carson and Renate Huber, (University of Western Ontario Series in Philosophy of Science)

Further reference:

M. Rédei, M. Stöltzner (eds.): *John von Neumann and the Foundations of Quantum Physics* (Vienna Circle Institute Yearbook 8), Kluwer, Dordrecht, 2001)

Structure of talk

- Two attitudes towards J. von Neumann's work
- Formal and soft axiomatism
- Opportunistic soft axiomatism in physics
- Opportunistic soft axiomatism at work in quantum mechanics
- Von Neumann on mathematical rigor
- Summary

Main claim of talk:

- Von Neumann had a very relaxed attitude about mathematical rigor **both** in mathematics **and** in physics
- A closer look at what von Neumann actually said about the scientific method and at how he actually acted in science, especially in quantum physics, shows that von Neumann's followed an **opportunistic soft axiomatism** in his work in physics.

Two typical attitudes towards
von Neumann's achievements in physics:

APPRECIATIVE :

great intellectual achievement
typical representative of successful
application of axiomatic method
in physics

AMBIVALENT:

great intellectual achievement **BUT**
example of useless and pointless
striving for mathematical exactness
in physics

To quote representatives of the appreciative camp:

The ‘axiomatic method’ is sometimes mentioned as the secret of von Neumann’s success. In his hands it was not pedantry but perception; he got to the root of the matter by concentrating on the basic properties (axioms) from which all else follows. The method, at the same time, revealed to him the steps to follow to get from the foundations to the application.

P. Halmos: *The legend of John von Neumann* American Mathematical Monthly **50** (1973) p. 394

“I do not know whether Hilbert regarded von Neumann’s book as the fulfillment of the axiomatic method applied to quantum mechanics, but, viewed from afar, that is the way it looks to me. In fact, in my opinion, it is the most important axiomatization of a physical theory up to this time.”

A. S. Wightman: Hilbert’s sixth problem, in *Proceedings of Symposia in Pure Mathematics* vol. 28,

AMS, 1976, p. 157

(Circumstantial) evidence for ambivalence:

“The mathematical rigor of great precision is not very useful in physics. But one should not criticize the mathematicians on this score ... They are doing their job.”

R. Feynman: *The character of physical law*, MIT Press, Cambridge, Mass. 1965 p. 56

“The maze of experimental facts which the physicist has to take into account is too manifold, their expansion too fast, and their aspect and relative weight too changeable for the axiomatic method to find a firm enough foothold, except in the thoroughly consolidated parts of our physical knowledge. Men like Einstein or Niels Bohr grope their way in the dark toward their conceptions of general relativity or atomic structure by another type of experience and imagination than those of the mathematician, although mathematics is an essential ingredient. Thus Hilbert’s vast plans in physics never matured.”

H. Weyl: “David Hilbert and his mathematical work” *Bulletin of The American Mathematical Society* **50** (1944)

One can distinguish two different notions of “axiomatising” and “axiomatic theory” in von Neumann’s works:

- axiomatising and axiomatic theory in the strict sense of formal systems or languages
(**formal axiomatics**)
- axiomatising and axiomatic theory in the less formal sense in which it occurs in physics
(**soft axiomatism**)

Formal axiomatics is what von Neumann does in his work on axiomatic set theory:

“We begin with describing the system to be axiomatized and with giving the axioms. This will be followed by a brief clarification of the meaning of the symbols and axioms It goes without saying that in axiomatic investigations as ours, expressions such as ‘meaning of a symbol’ or ‘meaning of an axiom’ should not be taken literally: these symbols and axioms do not have a meaning at all (in principle at least), they only represent (in more or less complete manner) certain concepts of the untenable ‘naive set theory’. Speaking of ‘meaning’ we always intend the meaning of the concepts taken from ‘naive set theory’”

J. von Neumann: Die Axiomatisierung der Mengenlehre, 1928, Collected Works vol. I p. 344, our translation

Soft axiomatism: a structured concept:
Von Neumann (Nordheim + Hilbert):

A physical theory consists of three parts:

- physical axioms
- analytic machinery (also called “formalism”)
- physical interpretation

- **Physical axioms**: semi-formal requirements (postulates) formulated for certain physical quantities and relations among them. The basis of these postulates is our experience and observations; thus the basis of the axioms in physics is empirical. (This is not necessarily the case in formal axiomatics: von Neumann points out that the fifth postulate in Euclid's geometry is non-empirical.)
- **Analytic machinery**: mathematical structure containing quantities that have the same relation among themselves as the relation between the physical quantities. **Ideally**, the physical axioms should be strong and rich enough to **determine** the analytic machinery **completely**.
- **Physical interpretation**: connects the elements of the analytic machinery and the physical axioms.

Soft axiomatism in (von Neumann, Nordheim and Hilbert)'s words
(formulated in case of QM)

“The way leading to this theory is the following: one formulates certain physical requirements concerning these probabilities, requirements that are plausible on the basis of our experiences and developments and which entail certain relations between these probabilities. Then one searches for a simple analytic machinery in which quantities appear that satisfy exactly these relations. This analytic machinery and the quantities occurring in it receive a physical interpretation on the basis of the physical requirements. The aim is to formulate the physical requirements in a way that is complete enough to determine the analytic machinery unambiguously.

This way is then the way of axiomatising, as this had been carried out in geometry for instance. The relations between geometric shapes such as point, line, plane are described by axioms, and then it is shown that these relations are satisfied by an analytic machinery namely linear equations. Thereby one can deduce geometric theorems from properties of the linear equations.

D. Hilbert, L. Nordheim, J. von Neumann: *Über die Grundlagen der Quantenmechanik* (1926)

Collected Works vol. I. p. 105, our translation

von Neumann (+ Hilbert and Nordheim):
not even soft axiomatism is practiced in actual science:

“In physics the axiomatic procedure alluded to above is not followed closely, however; here and as a rule the way to set up a new theory is the following.

One typically conjectures the analytic machinery before one has set up a complete system of axioms, and then one gets to setting up the basic physical relations only through the interpretation of the formalism. It is difficult to understand such a theory if these two things, the formalism and its physical interpretation, are not kept sharply apart. This separation should be performed here as clearly as possible although, corresponding to the current status of the theory, we do not want yet to establish a complete axiomatics.

What however is uniquely determined, is the analytic machinery which – as a mathematical entity – cannot be altered. What can be modified – and is likely to be modified in the future – is the physical

interpretation, which contains a certain freedom and arbitrariness.

D. Hilbert, L. Nordheim, J. von Neumann: Über die Grundlagen der Quantenmechanik (1926) p.

106, our translation

Opportunistic Soft Axiomatism

“I will further limit myself to saying a few things about procedure and method which will illustrate the general character of method in science. Not only for the sake of argument but also because I really believe it, I shall defend the thesis that the method in question is primarily opportunistic – also that outside of the sciences, few people appreciate how utterly opportunistic it is.”

J. von Neumann: *Method in the physical sciences* (1955), *Collected Works* vol. VI. p. 492

Opportunistic soft axiomatism at work in
von Neumann's 1926-1932 treatment of quantum mechanics

Only two, explicitly formulated **physical axioms**, both concern the nature of the expectation value of physical quantities a, b in a statistical ensemble:

A Expectation value assignments $a \mapsto E(a)$ are linear:

$$E(\alpha a + \beta b + \dots) = E(\alpha a) + E(\beta b) \dots$$

B Expectation value assignments are positive:

$$E(a) \geq 0 \quad \text{if } a \text{ can take on only non-negative values}$$

The two postulates spell out something that is a basic, empirically observable feature of expectation value assignments in a relative frequency interpreted probability theory

- **analytic machinery**: the set of **all** selfadjoint operators on a Hilbert space
- **physical interpretation**: specifying the bridge between the physical quantities and the operators:

C If the operators $A, B \dots$ represent the physical quantities $a, b \dots$ then the operator $\alpha A + \beta B + \dots$ represents the physical quantity $\alpha a + \beta b \dots$

D If operator A represents the physical quantity a then the operator $f(A)$ represents the physical quantity $f(a)$.

von Neumann:

$$A + B + C + D$$



every expectation value assignment is of the form

$$E(a) = \text{Tr}(UA) \tag{1}$$

with some statistical operator U (= positive, linear)

Formula (1) contains all probability statements in QM. Specifically:

$$\phi^U(P^Q(d)) = \text{Tr}(UP^Q(d))$$

is the probability of “quantum event” $P^Q(d)$ (projection) that the value of quantity Q lies in the Borel set d of real numbers in state ϕ^U . ($d \mapsto P^Q(d) \in \mathcal{P}(\mathcal{H}) = \text{spectral measure of } Q$).

Conceptual difficulty:

To interpret the non-commutative measure space $(\mathcal{P}(\mathcal{H}), \phi)$ as quantum probability space with probability understood as relative frequency (in von Mises' sense) the probability assignment $\mathcal{P}(\mathcal{H}) \ni Q \mapsto \phi(Q)$ needs to satisfy the **superadditivity** property:

$$\phi(X) + \phi(Y) = \phi(X \wedge Y) + \phi(X \vee Y) \quad (2)$$

Proposition: The superadditivity property is violated by **every** state ϕ for some projections.

Consequently: Lattice of projections **cannot** be interpreted as the structure representing random events for a quantum probability space if probability is interpreted as relative frequency

Von Neumann's **first** solution of the conceptual problem:

Rejecting Hilbert space quantum mechanics:

Birkhoff-von Neumann (1936):

The proper mathematical framework for quantum mechanics is the theory of type II_1 von Neumann algebras:

$$(\mathcal{P}(\mathcal{N}), d)$$

$\mathcal{P}(\mathcal{N}) =$ von Neumann lattice of projections
of type II_1 von Neumann algebra \mathcal{N}

$d =$ (relative) dimension function

(of Murray-von Neumann classification theory)

d is superadditive

“I would like to make a confession which may seem immoral: I do not believe absolutely in Hilbert space any more. After all Hilbert-space (as far as quantum-mechanical things are concerned) was obtained by generalizing Euclidean space, footing on the principle of “conserving the validity of all formal rules”. This is very clear, if you consider the axiomatic-geometric definition of Hilbert-space, where one simply takes Weyl’s axioms for a unitary-Euclidean-space, drops the condition on the existence of a finite linear basis, and replaces it by a minimum of topological assumptions (completeness + separability). Thus Hilbert-space is the straightforward generalization of Euclidean space, if one considers the *vectors* as the essential notions.

Now we [with F.J. Murray, von Neumann's coauthor] begin to believe, that it is not the *vectors* which matter but the *lattice of all linear (closed) subspaces*. Because:

1. The vectors ought to represent the physical *states*, but they do it redundantly, up to a complex factor, only.
2. And besides the *states* are merely a derived notion, the primitive (phenomenologically given) notion being the *qualities*, which correspond to the *linear closed subspaces*.

But if we wish to generalize the lattice of all linear closed subspaces from a Euclidean space to infinitely many dimensions, then one does not obtain Hilbert space, but that configuration, which Murray and I called “case II₁.” (The lattice of all linear closed subspaces of Hilbert-space is our “case I_∞”.) And this is chiefly due to the presence of the rule

$$a \leq c \rightarrow a \cup (b \cap c) = (a \cup b) \cap c$$

This “formal rule” would be lost, by passing to Hilbert space!”

Von Neumann's dream (around 1934-1935)

classical	quantum
classical logic	quantum logic
Boolean algebra	type II ₁ v. Neumann lattice
distributive	modular
random event structure	quantum (random) event structure
classical probability	quantum probability
normalized measure	dimension function d
relative frequency	relative frequency

Is $(\mathcal{P}(\mathcal{N}), d)$ with a type II_1 v. Neumann lattice
REALLY a NON-COMMUTATIVE probability space
whose probabilities can be interpreted as relative frequencies?

NO

because

Proposition (Murray-von Neumann) : d is the restriction to $\mathcal{P}(\mathcal{N})$
of (or can be extended to) a tracial state τ on \mathcal{N}

τ is a tracial state iff

$$\tau(XY) = \tau(YX) \quad \text{for all } X, Y \in \mathcal{N}$$

“ τ is insensitive for the non-commutativity”

Proposition : A linear functional on a von Neumann algebra is superadditive if and only if it is a trace

Consequently : $d(\tau)$ is the **ONLY** superadditive measure on $\mathcal{P}(\mathcal{N})$, so only those states satisfy a necessary condition for a relative frequency interpretation which disregard the non-commutative (non-classical) character of the “random event structure”

Therefore

if one wants to have a genuinely non-commutative probability space then the frequency interpretation has to go!

It did: von Neumann gave up the frequency view in 1937:

“This view, the so-called ‘frequency theory of probability’ has been very brilliantly upheld and expounded by R. von Mises. **This view, however, is not acceptable to us, at least not in the present ‘logical’ context.**”

“Quantum logic (strict- and probability logics)” unfinished, unpublished manuscript from 1937

Moral of von Neumann's treatment of quantum mechanics:

- Von Neumann's motivation in his research in physics was not the desire to create a mathematically impeccable axiomatic theory
(Hilbert space QM as von Neumann created it **was** mathematically rigorous)
- Von Neumann wanted to have a **conceptually clear and consistent** theory.
- Mathematical precision without conceptual-interpretational clarity in physics was for von Neumann neither possible nor desirable.

Von Neumann on mathematical rigor in pure mathematics

In his writings about pure mathematics and mathematical rigor von Neumann also takes a very down-to-earth, relaxed attitude concerning rigor:

“Whatever philosophical or epistemological preferences anyone may have in this respect, the mathematical fraternities’ actual experiences with its subject give little support to the assumption of the existence of an a priori concept of mathematical rigor.”

J. von Neumann: “The mathematician” (1947), in *Collected Works* vol. I. p. 6

“I have told the story of this controversy [*debate about the foundations of mathematics*] in such detail, because I think that it constitutes the best caution against taking the immovable rigor of mathematics too much for granted. This happened in our lifetime, and I know myself how humiliatingly easily my own views regarding the absolute mathematical truth changed during this episode, and how they changed three times in succession!”

J. von Neumann: “The mathematician” (1947), in *Collected Works* vol. I. p. 6

“... it is **not** necessarily true that the mathematical method is something absolute, which was revealed from on high, or which somehow, after we got hold of it, was evidently right and has stayed evidently right ever since. To be more precise, maybe it **was** evidently right after it was revealed, but it certainly didn't stay evidently right ever since. There have been very serious fluctuations in the professional opinion of mathematicians on what mathematical rigor is. To mention one minor thing: In my own experience, which extends over only some thirty years, it has fluctuated so considerably, that my personal and sincere conviction as to what mathematical rigor is, has changed at least twice. And this is in a short time of the life of one individual!”

J. von Neumann: “The role of mathematics in the sciences and in society” (1954) in *Collected Works* vol. VI. p. 480

Summary

- Mathematical rigor was for von Neumann a historically changeable concept.
- For a physical theory to be acceptable for von Neumann it had to be **both** mathematically reasonably precise **and** conceptually clear and consistent.
- Informal, conceptual inconsistency of a physical theory was for von Neumann sufficient reason to abandon that physical theory – even if it was mathematically rigorous (example: abandoning Hilbert space QM)
- In creating acceptable physical theories von Neumann was relying on the method of an opportunistically interpreted soft axiomatism.