The hidden variable problem in algebraic
relativistic quantum field theory

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Abstract

Given two quasilocal \( C^* \)-algebras \( \mathcal{A}, \mathcal{B} \) of relativistic quantum field theory, their state spaces \( E(\mathcal{A}), E(\mathcal{B}) \) and a positive, unit preserving map \( L: \mathcal{B} \to \mathcal{A} \) respecting the relativistic quasilocal structure of \( \mathcal{A} \) and \( \mathcal{B}, (\mathcal{B}, E(\mathcal{B})) \) is said to be a local hidden theory of \( (\mathcal{A}, E(\mathcal{A})) \) via \( L \) if for every state \( \phi \) in \( E(\mathcal{A}) \) the state \( L^* \phi \in E(\mathcal{B}) \) can be decomposed in \( E(\mathcal{B}) \) via a subcentral measure into states with pointwise strictly less dispersion than the dispersion of \( \phi \). It is shown that if there is a unique, locally normal, locally faithful, analytic vacuum state in \( E(\mathcal{A}) \) then \( (\mathcal{A}, E(\mathcal{A})) \) can not have a local hidden theory \( (\mathcal{B}, E(\mathcal{B})) \) via \( L \). This improves the result obtained in [J. Math. Phys. 28, 833 (1987)].

1 INTRODUCTION

In previous papers\(^1,2\) the problem of hidden variables was redefined in operator algebraic framework of quantum mechanics as the problem of finding conditions on the unital \( C^* \)-algebras \( \mathcal{A}, \mathcal{B} \) and on a positive, unit preserving map \( L: \mathcal{B} \to \mathcal{A} \) that imply that \( (\mathcal{B}, E(\mathcal{B})) \) is not a hidden theory of \( (\mathcal{A}, E(\mathcal{A})) \) via \( L \) [where \( E(\mathcal{A}), E(\mathcal{B}) \) are the state spaces of \( \mathcal{A} \) and \( \mathcal{B} \)]\(^3\) in the sense of the following definition.

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**Definition 1:** Here \((B, E(B))\) is a hidden theory of \((A, E(A))\) via \(L\) if for each state \(\phi \in E(A)\) one can find a positive (regular Borel) measure \(\mu\) on the state space \(E(B)\) with the help of which the composite state \(\phi \circ L = L^* \phi \in E(B)\) can be obtained in the integral form

\[
L^* \phi(x) = \int |\omega(x)| d\mu(\omega) \quad x \in B
\]

such that for all \(x \in B\) the following conditions hold

(2) \quad \text{if} \quad \sigma_\phi(x) > 0 \quad \text{then} \quad \sigma_\omega(x) < \sigma_\phi(Lx)

(3) \quad \text{if} \quad \sigma_\phi(x) = 0 \quad \text{then} \quad \sigma_\omega(x) = \sigma_\phi(Lx)

for all \(\omega \in \text{supp}(\mu)\), where \(\sigma_\phi(Lx) = \phi((Lx)^2) - \phi(Lx)^2\) and \(\sigma_\omega(x) = \omega(x^2) - \omega(x)^2\) and are the dispersions of the states \(\phi\) and \(\omega\), and \(\text{supp}(\mu)\) denotes the support set of \(\mu\).

This definition is a natural generalization of earlier hidden variable definitions, especially the one due to von Neumann. In view of this definition, if, besides having positivity and being unit preserving, \(L\) has certain additional algebraic properties, a negative result on the existence of a hidden theory of \((A, E(A))\) via \(L\) can be interpreted as determining an algebraic structure responsible for a given statistical uncertainty inherent in the description of physical systems by \(C^*\)-algebras that represent the algebraic structure in question. It can be proved, for instance, that \((B, E(B))\) is not a hidden theory of \((A, E(A))\) via a Jordan homomorphism \(L\) if \(A\) is a simple \(C^*\)-algebra, which means that if \((B, E(B))\) is a hidden theory of \((A, E(A))\) via \(L\) then the Jordan algebra structures of \(A\) and \(B\) must be regarded differently.

Based on Definition 1, the problem of local hidden variables also can be naturally reformulated in terms of quasilocai \(C^*\)-algebras: if both \(A\) and \(B\) are quasilocai \(C^*\)-algebras then \((B, E(B))\) is said to be a local hidden theory of \((A, E(A))\) via \(L\) if \(L\) preserves the local structure of \(A\) and \(B\) in some appropriate sense, \((B, E(B))\) is a hidden theory of \((A, E(A))\) in the sense of Definition 1, and the measure \(\mu\) in (1) can be chosen subcentral, which is to be interpreted as a natural locality property of \(\mu\).

A negative result on the existence of a local hidden theory of \((A, E(A))\) shows what locality properties must be violated if \((A, E(A))\) has a local hidden theory. It was proved in a previous paper that if \(L\) maps the local algebras onto local algebras in such a way that disjoint algebras are mapped onto disjoint ones and, in addition, \(L\) factorizes on disjoint algebras, then (modulo some technical assumptions) \((A, E(A))\) does not have a local hidden theory \((B, E(B))\) via \(L\).
provided the local algebras in both $\mathcal{A}$ and $\mathcal{B}$ are isomorphic to type I von Neumann algebras.\textsuperscript{8}

However, this result is unsatisfactory in at least two respects:

(i) the local algebras in relativistic quantum field theory can not be type I von Neumann algebras\textsuperscript{9} so the previous result does not apply to the relativistic case; and

(ii) even more importantly, due to the lack of expression of relativistic covariance on the quasi-local algebra, the locality properties of $L$ are not related to relativistic locality in any way. But it is just the relativistically local (also called Einstein local\textsuperscript{10}) hidden variable theories that have been the main subject of interest since Bell's work\textsuperscript{11}.

To have a result on the nonexistence of a local hidden theory of $(\mathcal{A}, E(\mathcal{A}))$, where $\mathcal{A}$ is a quasi-local $C^*$-algebra of relativistic quantum field theory is, furthermore, particularly desirable in light of the recent work of Summers and Werner, who have proved that Bell’s inequalities are maximally and typically violated in relativistic quantum field theory.\textsuperscript{12} The violation of Bell's inequalities in a theory $T$ is commonly interpreted in most papers in the literature on local hidden variables as the impossibility of $T$ being Einstein local. The results of Summers and Werner show that this interpretation is certainly not valid in relativistic quantum field theory, which implies that the usual approach to the local hidden variable problem through Bell’s inequalities does not work in relativistic quantum field theory.

In this paper we prove a proposition asserting the nonexistence of a local hidden theory $(\mathcal{B}, E(\mathcal{B}))$ of $(\mathcal{A}, E(\mathcal{A}))$ via $L$, where $\mathcal{A}, \mathcal{B}$, are quasi-local $C^*$-algebras of relativistic quantum field theory and $L$ is a positive, unit preserving map from $\mathcal{B}$ into $\mathcal{A}$ that has natural Einstein local properties (see Definition 2 below). Before stating the proposition we briefly recall the notion of quasi-local algebra in relativistic quantum field theory and define the Einstein local properties of $L$.

## 2 EINSTEIN LOCAL HIDDEN THEORIES

Recall that $\mathcal{A}$ is a quasi-local $C^*$-algebra of relativistic quantum field theory if $\mathcal{A}$ is the uniform closure of a net $\{\mathcal{A}(V), V \in M\}$ of (strictly) local $C^*$-algebras $\mathcal{A}(V)$ (with common unit) associated to the open, bounded subsets $V$ of the Minkowski space $M$, where the net has the following properties:

(i) if $V_1$ is contained in $V_2$ then $\mathcal{A}(V_1)$ is a subalgebra of $\mathcal{A}(V_2)$
(ii) if $V_1$ is spacelike separated from $V_2$ then every element of $\mathcal{A}(V_1)$ commutes with every element of $\mathcal{A}(V_2)$

(iii) there is a representation $\alpha$ of the identity connected component of the Poincaré group $P$ by automorphisms on $\mathcal{A}$ such that $\alpha_g(\mathcal{A}(V)) = \mathcal{A}(gV)$ for all $V$ and $g \in P$.

Part of the axioms of relativistic quantum field theory is also the assumption of existence of at least one physical representation of the quasi-local $C^*$-algebra $\mathcal{A}$, which means mathematically that one postulates the existence of an $\alpha$-invariant state $\phi \in E(\mathcal{A})$ (vacuum) such that the spectrum condition holds in the corresponding cyclic representation $\pi_{\phi}$.  

In what follows “relativistic quasilocal algebra” will always mean a quasi-local $C^*$-algebra of this type with the further assumption that all local algebras are von Neumann algebras and “relativistic quantum field theory” will mean a pair $(\mathcal{A}, E(\mathcal{A}))$ with a relativistic quasilocal algebra $\mathcal{A}$ and its state space $E(\mathcal{A})$, which is supposed to contain at least one vacuum state.

Let $\mathcal{A}$ and $\mathcal{B}$ be two relativistic quasilocal algebras and denote by $\beta$ the representation of $P$ on $\mathcal{B}$. We wish to define Einstein local properties of the positive, unit preserving map $L: \mathcal{B} \rightarrow \mathcal{A}$, by which we mean properties that express the similarity of the relativistic local structure of $\mathcal{A}$ and $\mathcal{B}$. In the following definition (a) is a natural locality demand, the content of (c) is that $L$ does not destroy the relativistic covariance whereas (b) is of technical nature.

**Definition 2:** The positive, unit preserving map $L: \mathcal{B} \rightarrow \mathcal{A}$ between two relativistic quasilocal algebras $\mathcal{A}$ and $\mathcal{B}$ is called Einstein local if

(a) $L$ maps the local algebras into local algebras,

(b) the restriction of $L$ to each local algebra is continuous in the ultraweak operator topology,

(c) $L$ commutes with the two representations $\alpha$ and $\beta$ in the sense that

\[(4)\quad \alpha_g \circ L = L \circ \beta_g \quad g \in P\]

We sum up with the following definition.

**Definition 3:** Let $\mathcal{A}$ and $\mathcal{B}$ be two relativistic quasilocal algebras. Here $(\mathcal{B}, E(\mathcal{B}))$ is a local hidden theory of $(\mathcal{A}, E(\mathcal{A}))$ if

(a) $L$ is Einstein local,

(b) $(\mathcal{B}, E(\mathcal{B}))$ is a hidden theory of $(\mathcal{A}, E(\mathcal{A}))$ via $L$ in the sense of Definition 1, and

(3) $\mu$ in (1) can be chosen subcentral.
Before formulating the proposition let us recall a few definitions and facts that will be used in the proof. A state \( \phi \) on \( \mathcal{A} \) is called locally normal if the restriction of \( \phi \) to every local von Neumann algebra \( \mathcal{A}(V) \) is ultraweakly continuous. The state \( \phi \) is said to be locally faithful if the condition \( x > 0 \) implies \( \phi(x) > 0 \) for any strictly local element \( x \in \mathcal{A}(V) \). Let \( T \) be the translation subgroup of \( P \). A state \( \phi \) is called a translation clustering state if

\[
\lim_{t \to \infty} \phi(x\alpha t y) = \phi(x)\phi(y)
\]

for all spacelike \( g \in T \) and for all local \( x, y \).

For a positive, unit preserving map \( L : \mathcal{B} \to \mathcal{A} \) the Cauchy-Schwartz inequality \( L(x^2) \geq L(x)^2 \) holds for all selfadjoint \( x \in \mathcal{B} \). Let \( \mu \) be a subcentral measure on \( E(\mathcal{B}) \) that decomposes a state \( \psi \in E(\mathcal{B}) \) in the sense of (1). Then if \( \psi \) is a factor state, i.e., if the center \( Z_\psi = \pi_\psi(B)^\sigma \cap \pi_\psi(B)^\sigma \) of the von Neumann algebra \( \pi_\psi(B)^\sigma \) generated by \( \psi \) in the Gelfand-Naimark-Segal (GNS) representation \( \pi_\psi \) consists of the multiples of the identity only, then \( \mu \) is the Dirac measure \( \delta_\psi \) concentrated at \( \psi \).

**Proposition 1:** Let \( (\mathcal{A}, E(\mathcal{A})) \) and \( (\mathcal{B}, E(\mathcal{B})) \) be two relativistic quantum field theories. If there is a non-dispersion-free, locally normal, locally faithful, \( \alpha \)-invariant, translation clustering state \( \phi \) on \( \mathcal{A} \) which is not dispersion-free then \( (\mathcal{A}, E(\mathcal{A})) \) does not have a local hidden theory via \( L \) in the sense of Definition 3.

**Proof:** The proof is similar to the proof of the Proposition in ref. 2. One shows first that if there is a state \( \psi \in E(\mathcal{A}) \) with nonzero dispersion \( \sigma_\psi(Lx_0) > 0 \) on some selfadjoint \( x_0 \in \mathcal{B} \) then \( (\mathcal{B}, E(\mathcal{B})) \) cannot be a hidden theory of \( (\mathcal{A}, E(\mathcal{A})) \) via \( L \) if the only measure that decomposes \( L^*\psi \in E(\mathcal{B}) \) is the Dirac measure \( \delta_{L^*\psi} \). To show this assume that \( (\mathcal{B}, E(\mathcal{B})) \) is a hidden theory of \( (\mathcal{A}, E(\mathcal{A})) \) via \( L \). By integrating (2) with respect to \( \mu = \delta_{L^*\psi} \) one obtains

\[
\psi((Lx_0)^2 - Lx_0^2) > \psi(Lx_0)^2 - \int \omega(x_0)d\mu(\omega) = 0
\]

which is a contradiction since the left hand side of (5) is not greater than zero by the Cauchy-Schwartz inequality for \( L \).

Thus to prove the Proposition 1 it is enough to show that there is a non-dispersion-free state \( \psi \in E(\mathcal{A}) \) such that \( L^*\psi \in E(\mathcal{B}) \) is a factor state. We prove that \( L^*\psi \) is a factor state over \( \mathcal{B} \) by showing that the assumption of \( L^*\phi \) not being a factor state contradicts the clustering property of \( \phi \).

Since \( \phi \) is \( \alpha \)-invariant \( L^*\phi \) is \( \beta \)-invariant by (3) of definition of Einstein locality of \( L \). Therefore both \( \alpha \) and \( \beta \) are implemented by
unitary representations $U$ and $W$ in the GNS representations $\pi_\phi$ and $\pi_{L^*\phi}$ of $A$ and $B$, respectively. Denote $L_\pi$ as the “representation” of $L$ in $\pi_\phi$ and $\pi_{L^*\phi}$, i.e. $L_\pi(\pi_{L^*\phi}(x)) = \pi_\phi(Lx)$. Then (c) in Definition 2 takes the form

\[ U_\pi L_\pi(\bullet) U_\pi^* = L_\pi W_\pi(\bullet) W_\pi^* \]

Assume that $L^*\phi$ is not a factor state. Then there is a non-trivial projector $\pi_{L^*\phi}(P)$ in the center

\[ Z = \pi_{L^*\phi}(B)^\prime \cap \pi_{L^*\phi}(B)^\prime \cap \pi_{L^*\phi}(B(V)) \cap \pi_{L^*\phi}(B(V)) \]

(The last inequality follows because $B$ is a quasilocal algebra.\textsuperscript{15}) By the local normality of $\phi$ and $L$, $L^*\phi$ is locally normal too and so $\pi_{L^*\phi}(B(V)) = \pi_{L^*\phi}(B(V))$. Thus $\pi_{L^*\phi}(P)$ is a nontrivial projector contained in (the center of) each local algebra $\pi_{L^*\phi}(B(V))$. Fix a $V$. Now $L$ maps the local algebras into local ones thus $L_\pi \pi_{L^*\phi}(P)$ is an element in some local algebra $\pi_\phi A(V')$. We may assume that $L_\pi \pi_{L^*\phi}(P)$ is nonzero if it were then we could take the orthogonal complement $\pi_{L^*\phi}(P)^\perp$, which shares all the properties of $\pi_{L^*\phi}(P)$ stated so far and the unit preserving property of $L$ implies that $L_\pi \pi_{L^*\phi}(P)$ and $L_\pi \pi_{L^*\phi}(P)^\perp$ can not both be zero. So $L_\pi \pi_{L^*\phi}(P)$ is a nonzero element, which is also positive by positivity of $L$ and this implies that $L_\pi \pi_{L^*\phi}(P)$ has a nontrivial spectral projector $L_\pi \pi_{L^*\phi}(e)$. By a theorem of Araki\textsuperscript{16} the elements of $Z$ commute with $W_\pi^*(g \in P)$ and so $L_\pi \pi_{L^*\phi}(P)$ commutes with $U_\pi^*(g \in P)$ by (6). But then $U_\pi^*$ commutes with $L_\pi \pi_{L^*\phi}(e)$, too, and it follows that $L_\pi \pi_{L^*\phi}(e)$ is a $U$-invariant nontrivial projector in $\pi_\phi A(V')$. Let $\pi_\phi(R) \in \pi_\phi A(V')$ be any nonzero local projector orthogonal to $L_\pi \pi_{L^*\phi}(e)$. Since $\phi$ is locally faithful we have

\[ \langle \Omega_\phi, \pi_\phi(R) \Omega_\phi \rangle > 0 \text{ and } \langle \Omega_\phi, L_\pi \pi_{L^*\phi}(e) \Omega_\phi \rangle > 0 \]

where $\Omega_\phi$ is the cyclic vector representing $\phi$ in the $\pi_\phi$ representation. On the other hand, by the clustering property of $\phi$ and by the $U$-invariance of $L_\pi L^*\phi(e)$,

\[ 0 = \langle \Omega_\phi, \pi_\phi(R) L_\pi \pi_{L^*\phi}(e) \Omega_\phi \rangle = \lim_{t \to \infty} \langle \Omega_\phi, \pi_\phi(R) U_\pi L_\pi \pi_{L^*\phi}(e) U_\pi^* \Omega_\phi \rangle = \langle \Omega_\phi, \pi_\phi(R) \Omega_\phi \rangle \langle \Omega_\phi L_\pi \pi_{L^*\phi}(e) \Omega_\phi \rangle \]

must hold too, which contradicts (7).
3 DISCUSSION

Let $\phi$ be a vacuum state in $E(A)$. If the vacuum vector $\Omega_\phi$ is analytic for the generator of time translations (in the representation $\pi_\phi$) then by the Reeh-Schlieder theorem\textsuperscript{17} $\phi$ is cyclic and separating for the local algebras, which means that in this case $\phi$ is locally faithful too. Obviously, the requirement that the vacuum state is not dispersion-free, is not a strong one, moreover, if the vacuum state $\phi$ in $E(A)$ is the unique $\alpha$-invariant state then $\phi$ is known to have also the translation clustering property (both in massive and in massless theories\textsuperscript{18}).

In this case, by (c) of Einstein locality of $L$, $L^*\phi$ is the unique $\beta$-invariant state, therefore by the assumption that $E(B)$ contains at least one vacuum state, $L^*\phi$ is the (unique) vacuum state in $E(B)$; in particular the spectrum condition is fulfilled in $\pi_{L^*\phi}$. So if $(A, E(A))$ is such that the vacuum state $\phi$ is a unique $\alpha$-invariant, analytic, locally normal state then the assumptions of Proposition 1 are fulfilled and Proposition 1 tells us that such $(A, E(A))$ relativistic quantum field theories are the best possible ones if “best” means “containing the least statistical uncertainty”, i.e., the statistical uncertainty inherent in the description of quantum fields by these theories cannot be reduced without violating either at least one of the Einstein local properties (1)-(3) in Definition 3 or the subcentrality of $\mu$.

Note that “statistical uncertainty” has been assumed in this paper being measured by the dispersion of quantum states. However, the dispersion is not the only conceivable measure of uncertainty of a probability distribution: the entropy also expresses a kind of uncertainty, which is conceptually different from what is expressed by dispersion. Based on the entropic measure of uncertainty the nonlocal hidden variable problem was reformulated in a previous paper\textsuperscript{19} and it will be interesting to investigate the local hidden variable problem in relativistic quantum field theory using entropic uncertainty.
References

3 For all the definitions and elementary facts in connection with the operator algebra theory we refer to O. Bratteli and D.W. Robinson, Operator Algebras and Quantum Statistical Mechanics (Springer, Berlin, 1979), Vol. 1.
4 See refs. 1 and 2 for a detailed motivation of this definition and its relation to previous ones.
5 Corollary 1 in ref. 1.
6 The theory of quasilocal $C^*$-algebras is described e.g. in Chap. 2.6 in ref. 3.
7 For the definition of subcentral measure see Ref. 3, p. 363, for a detailed explanation why subcentrality does express physical locality see Ref. 2. It should be noted that the locality (i.e. subcentrality) of the decomposing measure has nothing to do with the locality of $L$ or with the locality of the net of local algebras. The subcentrality can be – and is in fact – defined quite generally as a property of a measure on the state space of a general (unital) $C^*$-algebra.
8 See the Proposition in Ref. 2.
10 Here the informal notion “Einstein local theory” means a theory that complies with relativistic principles such as the principle of “no-action-at-a-distance”.
11 J.S. Bell, Physics (NY) 1, 195 (1961); J.S. Bell, in the Proceedings of the International School of Physics, “Enrico Fermi” Course 49 (Academic, Press, New York, 1971).
14 Proposition 3.2.4 in Ref. 3.
15 See Ref. 3, p. 122.