Von Neumann's mean ergodic theorem

- Physical background: foundational problem of classical statistical mechanics
- Notion of dynamical system
- Phase and time averages of phase functions
- The ergodic problem
- Koopman's Lemma, Koopmanism
- Von Neumann's and Birkhoff's ergodic theorems
- Status of ergodicity (metric transitivity)
- The friction between von Neumann and George Birkhoff about the discovery of ergodic theorems
- Comments, quotations on the ergodic problem

Main problem of CSM

Macroscopically large system $\mathcal{S}$ in "equilibrium"

F physical quantity of large system, constant in time

s_i $(i = 1, \ldots N)$ identical subsystems of $\mathcal{S}$ such that

– s_i's interact via known law

– s_i's states change in time according to known law

– N is "large" $(\sim 10^{23})$

How can F be viewed/expressed/computed as the average of a (suitable) function f that depends on the state's of s_i's?

Example:

Ideal gas

Conceptual Sketch of Standard Solution of Main Problem of CSM

One macroscopic equilibrium state $\leftrightarrow$ many microstates

$\Omega(E_1, E_2)$: # of (different) microstates between energies E_1, E_2

$$F = < f > = \int f \mathrm{d}p$$

f = appropriate phase function representing F

p = probability measure on phase space of large system expressing that all microstates compatible with a given macrostate are equally likely: the probability of a set of microstates is proportional to the size (measure) of the set:

$$p(\text{microstates between energies } \ E_1, E_2) \approx \Omega(E_1, E_2)$$

Principle of Equal Apriori Probability

Ω typically has a density ω:

$$\Omega(E_1, E_2) = \int_{E_1}^{E_2} \omega(x)\mathrm{d}x)$$

By the Principle of Equal Apriori Probability probabilities are proportional to the "number" (measure) of states

$$\Downarrow$$

The state density ω is a crucial quantity:

ω determines the probabilistic behavior of large system

But what justifies Principle of Equal Apriori Probability ?

sounds metaphysical

Can one justify the Principle of Equal Apriori Probability ?

Possible attitudes:

- The success of Principle justifies it, further justification is not needed/possible

  success:

  $\boxed{\text{(Principle + Assumptions)} \Rightarrow \text{empirically correct predictions}}$

- The Principle should/can be established by linking it to the properties of the dynamic of large system (determined by the dynamic of the subsystems)

  ergodic-type theorems

Main idea of justifying microcanonical measure by ergodic theory :

Separation of

microscopic (= short) and macroscopic (=long)

time scales

Macroscopic measurements yielding macroscopic quantity F

take place on macroscopic time scale

$\Downarrow$

F = long time average of microscopically evolving phase function f:

using microcanonical averaging would be justified if

$\boxed{\text{phase average of } f = F = \text{long time average of } f}$

Can the equality of phase average and time average be proved ?

$\Updownarrow$

The ergodic problem

The ergodic problem when

macrostate = energy of the system is fixed ($F = E = const.$)

Physical system = dynamical system

$$\Updownarrow$$

$$\langle X, \mathcal{S}, \mu, \{\alpha_t\} \rangle$$

- X = constant energy hypersurface of phase space of system
- $\{\alpha_t\}$ one parameter family of μ-preserving bijections on X
  $X \ni x \mapsto \alpha_t x \in X$ time evolution of system
- $f \colon X \to \mathbb{R}$ phase function, physical quantity

Principle of Equal Apriori Probability in $E = const.$ case:

The microstates compatible with $F = E = const.$

lie on the constant energy (hyper)surface X

the phase average of phase function f defined on X:

$$< f > = \int_X f \mathrm{d}\mu$$

where μ is the microcanonical probability measure:

$$\mu(A) = \frac{1}{v(X)} v(A)$$

with v being the surface measure on X

Ergodic problem:

Given the dynamical system

$\langle X, \mathcal{S}, \mu, \{T_t\} \rangle$

is the time average equal to the phase average?

$$\lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{+T} f(\alpha_t x)\mathrm{d}t \quad ?=? \quad \int_X f\,\mathrm{d}\mu$$

For the equality of time and phase averages to hold, the time average has to exist in the first place, and, since the time average involves a limit, one has to specify the topology in which the limit should exist.

Von Neumann's work on the problem concerned precisely this problem: he proved the existence of the time average with respect to a certain topology.

His specification of the topology and the proof of the theorem was strongly motivated by B.O. Koopman's idea of transforming the ergodic problem into one that is formulated in terms of operator theory:

"The possibility of applying Koopman's work to the proof of theorems like the ergodic theorem was suggested to me in a conversation with that author in the spring of 1930. In a conversation with A. Weil in the summer of 1931, a similar application was suggested, and I take this opportunity of thanking both mathematicians for the incentive which they furnished me for undertaking the investigations of this paper."

von Neumann: Proof or the quasi-ergodic hypothesis (1932)
Collected Works vol. II. No. 12

Proposition (Koopman's lemma): If $(X, \mathcal{S}, \mu)$ is a measure space and $T\colon X \to X$ is a measure preserving invertible function then the operator

$$U\colon L^2(X,\mu) \to L^2(X,\mu)$$

defined by

$$(Uf)(x) = f(Tx) \qquad\qquad \left[i.e. \quad Uf = f \circ T \ \right]$$

is a unitary operator

Sketch of proof: four steps:

- U is linear
- U is invertible
- U is an isometry on $L^1(X,\mu)$
- U is an isometry on $L^2(X,\mu)$

U is linear: for any complex numbers a, b

$$
\begin{aligned}
(U(af + bg))(x) &= (af + bg)(Tx) = \\
af(Tx) + bg(Tx) &= a(Uf)(x) + b(Ug)(x)
\end{aligned}
$$

U is invertible: The inverse U^{-1} of U is

$$(Vf)(x) = f(T^{-1}x)$$

U is an isometry on $L^1(X, \mu)$:

- U preserves the $\| \ \|_1$ norm of characteristic functions χ_A :

$$\begin{aligned} \|U\chi_A\|_1 = \|\chi_A \circ T\|_1 &= \|\chi_{T[A]}\|_1 = \mu(T[A]) = \\ = \mu(A) = \int_X \chi_A \mathrm{d}\mu &= \|\chi_A\|_1 \end{aligned}$$

- It follows that U preserves the $\| \ \|_1$ norm on finite linear combinations of characteristic functions (simple functions)

- If f is a positive function, then it is the pointwise limit of an increasing sequence f_n of non-negative simple functions $\Longrightarrow$ Uf_n also is an increasing sequence of non-negative functions $\Longrightarrow$ $\lim_n \|Uf_n\|_1 = \|Uf\|_1$ and $\lim_n \|f_n\|_1 = \|f\|_1$

- $\|f\|_1 = \||f|\|_1$ for any $f \in L^1(X, \mu)$

- U is an isometry on $L^2(X, \mu)$:

  The $\| \;\|_2$ and $\| \;\|_1$ norms are related as

$$\|f\|_2 = \sqrt{\|f^2\|_1}$$

QED of proof of Koopman's Lemma

Koopman's Lemma is the basis of the
technical trick to investigate the ergodic problem:
Koopmanism:

"Lift" the action of α_t from X to $L^2(X, \mu)$
and investigate the ergodic problem
in terms of functional analysis:

$$\begin{array}{rcl} L^2(X\mu) \ni f & \mapsto & U_t f \in L^2(X, \mu) \\ U_t f(x) & = & f(\alpha_t x) \end{array}$$

By Koopman's Lemma
$\{U_t\}$ is a unitary operator on $L^2(X, \mu)$ for every t

$t \longrightarrow U_t$ is a unitary representation
of the additive group of real numbers

Koopman:

If selfadjoint operator H is the generator of the representation:

$$U_t = e^{itH}$$

Then the ergodic (more generally: stochastic) behavior of dynamical system can be characterized by analyzing properties of the spectrum of H

"The pith of the idea in Koopman's method resides in the conception of the spectrum [of H] reflecting, in its structure, the properties of the dynamical system – more precisely, those properties of the system which are true 'almost everywhere' "

von Neumann: Proof or the quasi-ergodic hypothesis (1932)
Collected Works vol. II. No. 12

Proposition (Von Neumann's mean ergodic theorem) Given a dynamical system $\langle X, \mathcal{S}, \mu, \{\alpha_t\}\rangle$, the limit

$$f^*(x) := \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} f(\alpha_t x)\mathrm{d}t$$

exists in mean for all $f \in L^2(X, \mu)$

i.e. there exists an $f^* \in \text{Ł}^2(X, \mu)$ such that

$$\lim_{T\to\infty} ||f^* - \frac{1}{2T} \int_{-T}^{T} U_t f \mathrm{d}t|| = 0$$

recall: $\|f\| = \int_X f\bar{f}\mathrm{d}\mu$

Proposition [Mean Ergodic Theorem, discrete version]: If U is a unitary operator on a complex Hilbert space $\mathcal{H}$ and P is the projection on the subspace of $\mathcal{H}$ which are invariant under U, then

$$\lim_n \left[\frac{1}{n}\sum_{j=0}^{n-1} U^j f\right] = Pf$$

for every $f \in \mathcal{H}$

(The limit is in the sense of the Hilbert space norm !)

Idea of proof of (discrete) mean ergodic theorem:

Decompose every $f \in \mathcal{H}$ into two orthogonal components

$$f_P \qquad\qquad and \qquad\qquad f_{P^\perp}$$

- where P is the projection to the U invariant subspace, on which the convergence is obvious (since everything is left unchanged by U and hence by $\frac{1}{n}\sum_{j=0}^{n-1} U^j$ as well)
  and
- show that $\frac{1}{n}\sum_{j=0}^{n-1} U^j$ takes $f_{P^\perp}$ into zero (in the limit)

More detailed proof:

Claim: $P^{\perp}$ is the closure R of the linear subspace R_0 of elements of form $g - Ug$ for some $g \in \mathcal{H}$

To prove Claim one has to show:

- Elements of form $g - Ug$ is a linear subspace R_0 (trivial)
- $lim_n \left[\frac{1}{n} \sum_{j=0}^{n-1} U^j f \right] = 0$ for every $f \in R$
- $R^{\perp} = P$

Showing $lim_n \left[\frac{1}{n} \sum_{j=0}^{n-1} U^j f \right] = 0$ for every $f \in R$

Step 1. Assume $f \in R_0$, i.e. $f = g - Ug$ for some $g \in \mathcal{H}$, then

$$
\begin{aligned}
\sum_{j=0}^{n-1} U^j f &= \sum_{j=0}^{n-1} U^j \big(g - Ug \big) = \\
\sum_{j=0}^{n-1} U^j g &- \sum_{j=0}^{n-1} U^j U g = \\
U^0 g + \sum_{j=1}^{n-1} U^j g &- \sum_{j=1}^{n} U^j g = \\
&= g - U^n g
\end{aligned}
$$

$$\Downarrow$$

$$
\left\| \frac{1}{n} \sum_{j=0}^{n-1} U^j f \right\| = \frac{1}{n} \|g - U^n g\| \leq \frac{1}{n} \big(\|g\| + \|U^n g\| \big) = \frac{1}{n} 2 \|g\| \to 0
$$

Step 2. Assume $f \in R$ i.e. $f = \lim_n f_n$ for $f_n \in R_0$

Notational simplification: $Q_n = \frac{1}{n}\sum_{j=0}^{n-1} U^j$

We have $\|Q_n\| \leq \frac{1}{n}\|\sum_{j=0}^{n-1} U^j\| \leq \frac{1}{n}\sum_{j=0}^{n-1}\|U^j\| = 1$

Need to show: $Q_n f \to 0$ $(n \to \infty)$

$$\begin{aligned} \|Q_n f\| = \|Q_n f - Q_n f_k + Q_n f_k\| &\leq \|Q_n(f - f_k)\| + \|Q_n f_k\| \leq \\ \|Q_n\|\|(f - f_k)\| + \|Q_n f_k\| &\leq \|(f - f_k)\| + \|Q_n f_k\| \end{aligned}$$

Given $\epsilon > 0$ chose k large enough so that $\|(f - f_k)\| \leq \frac{\epsilon}{2}$, and since $\|Q_n f_k\| \to 0$ one can chose a large enough N so that $\|Q_n f_k\| \leq \frac{\epsilon}{2}$ for all $n > N$

Showing $R^{\perp} = P$

R_0 is dense in $R \Rightarrow R_0^{\perp} = R^{\perp}$, hence it is enough to see

If for $f \in \mathcal{H}$ it holds that

$\langle h, f \rangle = 0$ for every $h \in R_0$

then $f \in P$, i.e. $Uf = f$

$$\Updownarrow$$

(since $h \in R_0$ iff $h = g - Ug$ for $g \in \mathcal{H}$)

if $\langle (g - Ug), f \rangle = 0$ for every $g \in \mathcal{H}$ then $Uf = f$

$$\langle g, f \rangle - \langle Ug, f \rangle = 0 = \langle g, f \rangle - \langle g, U^* f \rangle = \langle g, (f - U^* f) \rangle$$

for all $g \in \mathcal{H}$

$$\Downarrow$$

$$f - U^* f = 0$$

$$\Updownarrow$$

$$U^* f = f$$

$$\Updownarrow$$

$$Uf = f$$

Proposition (Birkhoff's pointwise ergodic theorem) Given a dynamical system $\langle X, \mathcal{S}, \mu, \{\alpha_t\}\rangle$ the limit

$$f^*(x) := \lim_{T\to\infty} \frac{1}{2T}\int_{-T}^{T} f(\alpha_t x)\mathrm{d}t$$

exists for μ-almost every $x \in X$ and for all $f \in L^1(X, \mu)$

Definition : The dynamical system $\langle X, \mathcal{S}, \mu \rangle$ is metrically transitive if the α invariant measurable sets have measure 0 or 1, i.e.

$$\text{if}$$
$$\Big[A \in \mathcal{S} \qquad \textit{and} \qquad \alpha_t[A] \subseteq A \Big]$$
$$\text{implies}$$
$$\mu(A) = 0 \qquad \textit{or} \qquad \mu(A) = 1$$

Metrically transitive dynamical systems
are also called
ergodic dynamical systems

Definition: $f\colon X \to \mathbb{R}$ is called α invariant if

$$f(\alpha_t x) = f(x) \qquad\qquad \text{for } \mu \text{ almost all } x \in X$$

Proposition: For a dynamical system $\langle X, \mathcal{S}, \mu, \{\alpha_t\}\rangle$ the following are equivalent:

1. the system is ergodic
2. if f is a measurable α invariant function on X then f is constant a.e.

Proof:

2. $\Rightarrow$ 1. By contraposition: If the system is not ergodic, then there exists an α invariant measurable set A such that $\mu(A) \neq 0, 1$, hence

$$A \neq X, \emptyset \qquad \text{and} \qquad A^{\perp} \neq X, \emptyset$$

and the function

$$f = \chi_A - \chi_{A^{\perp}}$$

is measurable, α invariant but not constant a.e.

1. $\Rightarrow$ 2. Assume f is α invariant but not constant a.e. Then there exist $x_1, x_2 \in X$ such that $f(x_1) < f(x_2)$ and such that none of the following disjoint inverse image measurable sets

$$\begin{aligned} f^{-1}\big((-\infty, f(x_1)]\big) &\subseteq X \\ f^{-1}\big((f(x_1), \infty)\big) &\subseteq X \end{aligned}$$

can have zero μ measure. For if any were zero, then one could change f on a set of zero measure in such a way that $f(x_1) = f(x_2)$. Indeed: if, say

$$\mu\Big(f^{-1}\big((-\infty, f(x_1)]\big)\Big) = 0$$

then defining

$$f'(x) = \begin{cases} f(x), & x \in f^{-1}\big((f(x_1), \infty)\big); \\ f(x_2), & x \in f^{-1}\big((-\infty, f(x_1)]\big). \end{cases}$$

one would have $f = f'$ a.e. and $f'(x_1) = f'(x_2)$.

Since f is invariant, the disjoint sets

$$\begin{aligned} f^{-1}\big((-\infty, f(x_1)]\big) &\subseteq X \\ f^{-1}\big((f(x_1), \infty)\big) &\subseteq X \end{aligned}$$

are also invariant, so by ergodicity of the system

$$\begin{aligned} \mu\Big(f^{-1}\big((-\infty, f(x_1)]\big)\Big) &= 0 \textit{ or } 1 \\ \mu\Big(f^{-1}\big((f(x_1), \infty)\big)\Big) &= 0 \textit{ or } 1 \end{aligned}$$

Since none of the above measures can be equal to 0, both have to be equal to 1, which is a contradiction since the sets are disjoint and the sum of their measures has to be less than 1

QED

$\langle X, \mathcal{S}, \mu, \{\alpha_t\}\rangle$ is ergodic

$+$

von Neumann's

Mean Ergodic Theorem

$\Downarrow$

time average of $f \in L^2$

(in the mean $\| \ \|_2$)

is constant

$\Downarrow$

is equal to

the phase average

Birkhoff's

Pointwise Ergodic Theorem

$\Downarrow$

time average of $f \in L^1$

(pointvise)

is constant

$\Downarrow$

is equal to

the phase average

Thus if the dynamical system $\langle X, \mathcal{S}, \mu, \{\alpha_t\}\rangle$
on the constant energy hypersurface X
defined by the dynamics of the subsystems in CSM
is metrically transitive (ergodic) then the
Principle of Equal Apriori Probability
is justified

But it is justified in two different senses:
according to the two different notions of convergence
(in mean and pointwise)

"In view of these facts, it is of interest to decide which of the two formulations [convergence in mean or pointwise convergence] corresponds to the actual physical problem of the ergodic hypothesis. It turns out that the weaker form of statement [convergence in mean] is sufficient, – that it, indeed is the precise mathematical equivalent of the physical state of affairs."

von Neumann: Physical application of the ergodic hypothesis (1932)
in Collected works vol. II. No. 13

Von Neumann's argument:

If one wants the time average $\frac{1}{t-s}\int_s^t f(\alpha_t x)dt$ to be "physically replaceable" by the phase average (a constant), one wants the time average to be replaceable by a constant C such that

" ... the statistical dispersion of $\frac{1}{t-s}\int_s^t f(\alpha_t x)dt$ about C is small, i.e.

$$\int_X \left| \frac{1}{t-s}\int_s^t f(\alpha_t x)dt - C \right|^2 d\mu(x) < \epsilon$$

for a given ϵ. Now the [convergence in mean] states precisely that this is the case for $t-s$ sufficiently large ..."

von Neumann: Physical application of the ergodic hypothesis (1932)
in Collected works vol. II. No. 13

In view of significance of ergodicity it is crucial to know:

Are the dynamical systems $\langle X, \mathcal{S}, \mu, \{\alpha_t\} \rangle$

(at least the ones occurring as models of physical systems)

metrically transitive (ergodic)?

More generally:

On what conditions is a dynamical system ergodic?

Extremely difficult problem !!

interesting in its own right

important in many branches of mathematics

(not only in CSM)

Note that

- von Neumann did NOT prove ergodicity (i.e. metric transitivity) of any dynamical system (specifically of a dynamical system of physical interest)
- hence the title of Von Neumann's paper

  "Proof of the quasi-ergodic hypothesis"

  is very misleading because quasi-ergodicity for von Neumann is synonym with ergodicity (i.e. with metric indecomposability)

The title of Von Neumann's paper

"Proof of the quasi-ergodic hypothesis"

is very misleading for another reason as well:

"quasi-ergodicity" is sometimes used to refer to the following

topological transitivity property of dynamical systems:

Definition: If X is a topological space then the dynamical system $\langle X, \mathcal{S}, \alpha, \mu \rangle$ is quasi-ergodic if the trajectory of μ almost every point in X is topologically dense in X

Quasi ergodicity in this topological sense differs from ergodicity (metric transitivity):

- ergodicity concerns just the measure theoretic aspects of the dynamic (X need not even have any topology for ergodicity to make sense).
- Even if X does have a topology, quasi-ergodicity in the topological sense does not imply ergodicity (metric transitivity) in general, not even if X is a compact metric space
- The converse is true, however:
  Proposition: If X is a (compact) topological space and μ is a Borel measure on X such that $\mu(A) > 0$ for every open set A, then ergodicity implies that the orbit $\{\alpha_t x\}$ of μ almost every x is everywhere dense in X

The status of ergodicity of dynamical systems occurring in CSM is still to a large extent an open problem

- It has been claimed (Y. Sinai) that hard spheres in a box with elastic collisions as the only interaction is ergodic but the full proof of this claim has never been published and the claim is considered as (yet) unproven
- Results on the general theory of dynamical systems indicate that ergodicity is not a property dynamical systems typically possess

Even if ergodicity could be established for dynamical systems in physics, the explanation of the Principle of Equal Apriori Probability would only be an explanation
up to measure zero set

$\Downarrow$

we seem to be forced to assume that
the system is not in fact in the measure zero set

"Revenge of the measure zero set"

Attempts have been made to weaken the notion of ergodicity:

- Requiring ϵ-ergodicity only
  The dynamical system is $0 < \epsilon$-ergodic if it is metrically transitive on an invariant set of measure $(1-\epsilon)$: there is a set $X' \subseteq X$ of measure $(1-\epsilon)$ such that if $A \subseteq X'$ and $T_t[A] \subseteq A$ then $\mu(A) = 0$ or $\mu(A) = 1 - \epsilon$ (Vranas, 1998)
  The status of epsilon ergodicity is unclear
- Requiring equality of phase and time averages for the special phase functions that are sums of phase functions of subsystems (Khinchin, 1949)

Khinchin's weakening of ergodicity
(using asymptotic form of energy density/microcanonical measure)

Proposition

$$\mu\left(\left\{s \in X : \left|\frac{\langle f(s)\rangle_T - \langle f\rangle_\mu}{\langle f\rangle_\mu}\right| \geq K_1 N^{-1/4}\right\}\right) \leq K_2 N^{-1/4}$$

μ = microcanonical measure

$K_1, K_2 > 0$ constants

$f = \sum_{i=1}^{N} f_i$ phase function = sum of phase functions of subsystems

The set of points on the constant energy hypersurface where the time average of a sum function differs from its microcanonical average more than some amount that goes to zero as $N \to \infty$ has a measure that also goes to zero as $N \to \infty$.

Revenge of the measure zero set $\Rightarrow$ Terror of the small measure set

“Today’s formulation of the ergodic theorem is that, except for a set of measure 0, the time average exists and = phase average. All the hypothesis of ‘disorderliness’ are contained in the assumption that we are not in that measure zero set.”

“A mai fogalmazása az ergodikus tételnek az, hogy egy 0-measure-ű halmaztól eltekintve a Zeitmittelwert egy *létező* limes és = Raummittelwert. Az összes “rendezetlenségi” hipotézisek bennefoglaltatnak abban a feltevésben, hogy *nem* vagyunk abban a 0-measure-ű halmazban.”

J. von Neumann to R. Ortvay (February 2, 1939)

Princeton

January 16, 1932

Dear Robertson:

The story of the proof of the quasi-Ergodic hypothesis, about which you were asking me, is this:

Koopman informed me of his method, which is now published (Proc. Nat. Ac., May 1931), and well known, to you, during my first stay in this country, in May 1930. He expressed his hope to prove the q.E.h by means of it, but none of us succeeded then. Andre Weil, who lives usually in Aligarh, India, came to Berlin in Summer 1931, and visited me there. He told me about a method, he worked out, to attack dynamical problems, and hoped to solve the q.E. problem, but he did not succeed with it till then either. This was in July 1931 (since then Weil wrote letters to Koopman and to me, acknowledgeing Koopman's priority).

In August-September 1931 I took this problem up again, and solved it, in the way known to you. I did not publish it, because I wanted to know first, if Koopman had not obtained the same result meantime. I would have considered it as dishonest, if I had interfered with his investigations, using his methods, without his authorization.

In October 1931, I met Koopman in New York, and he informed me, that my result was absolutely new to him. He recommended me, to publish it in the Proc.Nat.Ac., where his method has been published, too. I wrote my paper then, and gave it to him, as he kindly accepted to take care of its translation.

At the New York meeting of the American Math. Soc. (October) M. Stone and E. Hopf saw my manuscript, and we discussed it. At the inauguration of Fine Hall in Princeton (October 1931), Koopman and Birkhoff [Georg David Birkhoff, not to be confused with Garrett Birkhoff] came here, and Koopman and myself informed Birkhoff about the fact, that I had proven the q.E.h. with Koopman's method, and told him the exact statement. He then learned, what I considered my essential results:

1. that the limit of the "relativ sojourn" (in the time interval $s < \tau < t$, for $t - s \to +\infty$ *allways* exists, if it is understood as convergence in the mean,
2. that the q.E.h is true *if and only if* there are no measurable integrals.

In November 1931 Birkhoff wrote me a letter, congratulating me for this proof; informing me, that he wanted to talk about this new situation in his general talk in New Orleans; asking me to write him the exact statement of my result; and informing me, that he had a direct method to attack the problem, and hoped to succeed with it.

On December 4/5, 1931 several Princeton men, between them myself went to Cambridge, to give and heare conferences in theoretical physics. I gave there my proof, and in the discussions Birkhoff informed us, that he had another proof, which showed even somewhat more than mine: instead of mean-conv. he could prove conv. everywhere except on a set of measure 0. (In a paper to appear in the Proc.Nat.Ac. [J. von Neumann: "Physical applications of the ergodic hypothesis" Collected

Works Vol. II. No. 13.] I show, that the physical statement of the q.E.h. requires mean-conv., and not more.) On the dinner at the Harvard Club, which was given by Harvard and Mass.Tech. faculties to us, I asked Birkhoff, when his paper will appeare, as mine would come out in the January issue of the Proc.Nat.Ac. He told me, that he tried to get his into the December number. I asked him to withhold, for obvious reasons, but he refused that, stateing moreover, that it was dubious whether it will come out already in December. As this was not the place to argue, I did not insist, he said besides, that he will acknowledge my priority in due form. (I did *not* agree to his quick-publishing, I only gave up on this subject of talk).

Some days before I went to Cambridge, E. Hopf informed me, that he had a new proof of the q.E.h., simpler than mine. When I met him in Cambridge, he told me, that his first attempt contained an error. He had another proof-variant, but this essentially followed my line of argument, only avoiding use of the "spectral resolution" for Koopman operators U_t. He wanted to publish it in the Proc.Nat.Ac. too, and had given his manuscript to Birkhoff, he naturally agreed with me that it should not come out earlier than mine.

(My paper was delayed to some extent by the fact, that Koopman discussed it with M. Stone, who advised some technical changes, which are very interesting, and of one of which I made use. So the manuscript was only sent in on December 9, too late for the December number).

The December number appeared in January 1932, and Birkhoff's article was in it. His quotation of my result is, according to the judgement of Eisenhart, Alexander, Lefschetz, Koopman, Stone, absolutely insufficient. (I mention only these 5 names, because they all spontaneously told me, how dissatisfied they are, without any attempt on my side, to talk about this matter.) The reason they give is, that it does not show to any person, uninformed about the real history of these things, who of Birkhoff and myself got the other started; that which one of us attacked the unsolved q.E.h, and which one found an independent proof, after he knew that it was solved, and what the necessary and sufficient conditions for its truth are.

Excuse me, for boring you with the lengthy and teadious details of the story, but you wanted me to inform you reliably and exactly.

With the best greetings Your,

J. v. Neumann

Original in the Institute Archives, California Institute of Technology, language of original: English. Published in full first in J.D. Zund: "Georg David Birkhoff and John von Neumann: A question of priority and the ergodic theorems, 1931-1932, *Historia Mathematica* **29** (2002) 138-156

New York

Jan. 22, 1932

Dear Mr. Stone,

Koopman informs me, that you took an interest in my affair with B.[irkhoff]. I thank you heartily for this. He asks me to inform you about my talks and correspondence with B.[irkhoff] in this matter. They where as follows:

1. B.[irkhoff] was told in Princeton, at the inauguration of Fine Hall (Oct. 22) by Koopman and myself that I proved with K.[oopman]s method this: the q.[uasi] E.[rgodic] Theorem is true if and only if there are no measurable integrals, and that the relative-sojourn limit allways exists. [Von Neumann's insertion:] "We told him, that it was 'in the mean', but he may not have paid attention to it".
2. B.[irkhoff] wrote me a letter sometime in November, expressing his satisfaction about these results, asking me for an exact statement of them, for two reasons: because he wanted to refer to them in New

Orleans, and because he *hoped* to prove them directly.
I answered him immediately, giving the exact statement of conv.[ergence] in the mean for all motions, and the nec.[essary] and suff.[icient] conditions for the q.[uasi] E.[rgodic] hypothesis.

3. In Cambridge (Dec. 5) I gave a talk on my results. In the discussion B.[irkhoff] stated, that he had proved somewhat more: conv.[ergence] excepted on a 0-measure set.

4. On the same evening, at a dinner given by the faculties, I asked him, when his paper would appear. He said: Proc.[eedings of the] Nat.[ional] Ac.[ademy of Sciences], December of January. I asked him to wait, till mine would appear. He refused this, saying: Nobody likes to delay his own papers, and I will state your priority in due form. I gave up the conversation at this point, as I could not force him to act as I would have expected – naturally without a word which could have been interpreted as my accepting his view. –

Hoping to hear from you soon, I am truly yours

John von Neumann.

Budapest, (Hungary),

V. Arany János utca 16

Quotations

"This fundamental postulate [Principle of Equal Apriori Probability] is eminently reasonable and certainly does not contradict any of the laws of mechanics. Whether the postulate is actually valid can, of course, only be decided by making theoretical predictions based on it and by checking whether these predictions are confirmed by experimental observations. A large body of calculations based on this postulate have indeed yielded results in very good agreement with observations. The validity of this postulate can therefore be accepted with great confidence as the basis of our theory."

F. Reif: Fundamentals of statistical and thermal physis (McGraw-Hill, 1965) p. 55

"... in all expositions of the statistical mechanics, this phase average is taken as a theoretical interpretation of any physical quantity. In doing so either no arguments at all are given in favor of such a choice, or a special hypothesis is constructed in order to justify this choice, or, finally, various reasons are cited in favor of such an interpretation, indicating at the same time that these reasons are not logically obligatory and that the interpretation was generally accepted in view of the successful results to which the theory based on this interpretation leads."
A.I. Khinchin: Mathematical Foundations of Statistical Mechanics
(Dover Publications, 1948) p. 46

"...the task of a mathematical justification of the statistical mechanics reduces essentially to two problems. The first of these two problems, to investigate as exhaustively as possible, under what conditions and to what degree the time averages of phase functions, which, as we have seen, appear as a natural interpretation of experimental measurements, can be replaced by the phase averages of the same functions. ... The second problem ... is to create a general method for approximate computation of phase averages on surfaces of constant energy."

A.I. Khinchin: Mathematical Foundations of Statistical Mechanics (Dover Publications) 1948, p. 47

“We emphasize once more that this distribution [the microcanonical distribution] is not the genuine statistical distribution of a closed system. If it were, then this would be equivalent to the claim that, during a sufficiently long time, the phase trajectory of a closed system would come arbitrary close to any point of the manifold [of the constant energy surface]. This claim (under the name ergodic hypothesis) is however false in general.”

L.D. Landau, E.M. Lifschitz: Lehrbuch der Theoretischen Physik V. Statistische Physik (Akademie Verlag, 1971) p. 13 [my translation]

"The physical importance of ergodicity is that it can be used to justify the use of the microcanonical ensemble for calculating equilibrium values and fluctuations. Suppose f is some macroscopic observable and the system is started at time zero from a dynamical state x, for which $f(x)$ has a value that is very far from its equilibrium value. As time proceeds, we expect that the current value of f, which is $f(T_t x)$, will approach and mostly stay very close to an equilibrium value with only very rare large fluctuations away from this value. This equilibrium value should therefore be equal to the time average because the initial period during which equilibrium is established contributes only negligibly to the formula defining $f^*(x)$. The [ergodic] theorem tells us that this equilibrium value is almost equal to $\langle f \rangle$, the average value of f in the microcanonical ensemble, provided the system is ergodic."

L. Lebowitz and R. Penrose: Modern ergodic theory Physics Today, vol. 26 (1973)

“Strict ergodicity has turned out to be surprisingly difficult to prove even for relatively simple dynamical systems. Contrary to what is sometimes asserted, the system of N elastic hard balls moving in a cubical box with hard reflecting walls has not yet been proven to be ergodic for arbitrary N – only for $N \leq 4$ [references] Nevertheless it iseems that mathematicians are coming increasingly closer to a proof [references], and computational evidence suggests that this system is indeed ergodic [references].”

“... epsilon-ergodicity needs to be investigated in more detail. It would be nice to have theoretical results showing both that ergodicity is approached when the number of degrees of freedom increases and that ϵ decreases faster than exponentially with the number of degrees of freedom.”

P. Vranas: Epsilon-ergodicity and the success of equilibrium statistical mechanics Philosophy of Science vol. 65 (1998)

Summary

- The Principle of Equal Apriori Probability is crucial in classical (equilibrium) statistical mechanics: This principle is used to determine the probability distributions utilized to compute averages of phase functions.
- Ergodic theory is an attempt to justify the Principle of Equal Apriori Probability by linking the apriori probability measure to the dynamics of the many body system
- Von neumann's mean ergodic theorem was a decisive breakthrough in ergodic theory because:
  – it proved that the time averages exist (in mean)
  – isolated ergodicity (metric transitivity) as the crucial property of dynamical systems needed for foundational purposes
- Attempts to justify the Principle of Equal Apriori Probability via ergodic-type theorems by proving metric transitivity are inconclusive, with unsolved difficult problems

- "Revenge" of measure zero sets remains a problem even if ergodicity obtains