The (negation) incompleteness of the first-order theory of canonical calculi and the unprovability of consistency

András Máté

19.05.2023

Diagonalization in **CC** (preparatory steps)

Diagonalization in **CC** (preparatory steps)

Lemma 1.: the true *closed atomic formulas* of $\mathcal{L}^{1*}$ resp. $\mathcal{L}^{10}$ are all provable in $\mathbf{CC}^*$ resp. in **CC**.

Diagonalization in **CC** (preparatory steps)

Lemma 1.: the true *closed atomic formulas* of $\mathcal{L}^{1*}$ resp. $\mathcal{L}^{10}$ are all provable in **CC*** resp. in **CC**.

The theorems of **CC*** can be defined by the calculus Σ^* and the theorems of **CC** by Σ. They contain some auxiliary letters to describe the language of **CC*** resp. of **CC**, e.g. **F** for formula and **S** for substitution. (Boldface is used to distinguish auxiliary letters of these calculi from the auxiliary letters of the hypercalculi.)

Diagonalization in **CC** (preparatory steps)

Lemma 1.: the true *closed atomic formulas* of $\mathcal{L}^{1*}$ resp. $\mathcal{L}^{10}$ are all provable in $\mathbf{CC}^*$ resp. in **CC**.

The theorems of $\mathbf{CC}^*$ can be defined by the calculus Σ^* and the theorems of **CC** by Σ. They contain some auxiliary letters to describe the language of $\mathbf{CC}^*$ resp. of **CC**, e.g. **F** for formula and **S** for substitution. (Boldface is used to distinguish auxiliary letters of these calculi from the auxiliary letters of the hypercalculi.)

In constructing the hypercalculi $\mathbf{H}_2$ and $\mathbf{H}_3$ we have used an encoding procedure. Let us denote the code of a string f by $[f]'$ (the square brackets can be omitted if f consists of a single letter or a metavariable). Be $[\Sigma^*]' = \sigma^*$and $\Sigma' = \sigma$.

Diagonalization in **CC** (preparatory steps)

Lemma 1.: the true *closed atomic formulas* of $\mathcal{L}^{1*}$ resp. $\mathcal{L}^{10}$ are all provable in $\mathbf{CC}^*$ resp. in **CC**.

The theorems of $\mathbf{CC}^*$ can be defined by the calculus Σ^* and the theorems of **CC** by Σ. They contain some auxiliary letters to describe the language of $\mathbf{CC}^*$ resp. of **CC**, e.g. **F** for formula and **S** for substitution. (Boldface is used to distinguish auxiliary letters of these calculi from the auxiliary letters of the hypercalculi.)

In constructing the hypercalculi $\mathbf{H}_2$ and $\mathbf{H}_3$ we have used an encoding procedure. Let us denote the code of a string f by $[f]'$ (the square brackets can be omitted if f consists of a single letter or a metavariable). Be $[\Sigma^*]' = \sigma^*$and $\Sigma' = \sigma$.

Lemma 2.: If a string f is derivable in Σ, then $\sigma D f'$ is derivable in $\mathbf{H}_2$. Therefore, $D(\sigma)(f')$ is a true atomic formula of $\mathcal{L}^{10}$. According to Lemma 1., it is a theorem of **CC**.

The diagonal formula

The diagonal formula

Be A a formula with at most one variable x of the language $\mathcal{L}^{10}$, $A' = a$, $B = A^{a/x}$ and $B' = b$.

The diagonal formula

Be A a formula with at most one variable x of the language $\mathcal{L}^{10}$, $A' = a$, $B = A^{a/x}$ and $B' = b$.

Let us assume first that $\Gamma_0 \vdash B$. In this case, Σ derives the words $\mathbf{F}A$, $B\mathbf{S}A\mathbf{S}a\mathbf{S}x$, B.

The diagonal formula

Be A a formula with at most one variable x of the language $\mathcal{L}^{10}$, $A' = a$, $B = A^{a/x}$ and $B' = b$.

Let us assume first that $\Gamma_0 \vdash B$. In this case, Σ derives the words $\mathbf{F}A,\ B\mathbf{S}A\mathbf{S}a\mathbf{S}x,\ B$.

Therefore by Lemma 2., the following atomic formulas are theorems of **CC**: $D(\sigma)(\mathbf{F}'a),\ D(\sigma)(b\mathbf{S}'a\mathbf{S}'a'\mathbf{S}'x'),\ D(\sigma)(b)$.

The diagonal formula

Be A a formula with at most one variable x of the language $\mathcal{L}^{10}$, $A' = a$, $B = A^{a/x}$ and $B' = b$.

Let us assume first that $\Gamma_0 \vdash B$. In this case, Σ derives the words $\mathbf{F}A$, $B\mathbf{S}A\mathbf{S}a\mathbf{S}x$, B.

Therefore by Lemma 2., the following atomic formulas are theorems of **CC**: $D(\sigma)(\mathbf{F}'a)$, $D(\sigma)(b\mathbf{S}'a\mathbf{S}'a'\mathbf{S}'x')$, $D(\sigma)(b)$.

Let us abbreviate their conjunction by $Diag_\sigma(a, b)$. If b was a theorem in **CC**, then this diagonal formula is a theorem, too.

The diagonal formula

Be A a formula with at most one variable x of the language $\mathcal{L}^{10}$, $A' = a$, $B = A^{a/x}$ and $B' = b$.

Let us assume first that $\Gamma_0 \vdash B$. In this case, Σ derives the words $\mathbf{F}A$, $B\mathbf{S}A\mathbf{S}a\mathbf{S}x$, B.

Therefore by Lemma 2., the following atomic formulas are theorems of **CC**: $D(\sigma)(\mathbf{F}'a)$, $D(\sigma)(b\mathbf{S}'a\mathbf{S}'a'\mathbf{S}'x')$, $D(\sigma)(b)$.

Let us abbreviate their conjunction by $Diag_\sigma(a, b)$. If b was a theorem in **CC**, then this diagonal formula is a theorem, too.

Let us now assume that $Diag_\sigma(a, b)$ is a theorem. Then each conjunct is a theorem, too, so they are true according our truth definition. The third conjunct says that the calculus with the code σ derives the string with the code b, i.e., B is a theorem of **CC**.

The diagonal formula

Be A a formula with at most one variable x of the language $\mathcal{L}^{10}$, $A' = a$, $B = A^{a/x}$ and $B' = b$.

Let us assume first that $\Gamma_0 \vdash B$. In this case, Σ derives the words $\mathbf{F}A$, $B\mathbf{S}A\mathbf{S}a\mathbf{S}x$, B.

Therefore by Lemma 2., the following atomic formulas are theorems of **CC**: $D(\sigma)(\mathbf{F}'a)$, $D(\sigma)(b\mathbf{S}'a\mathbf{S}'a'\mathbf{S}'x')$, $D(\sigma)(b)$.

Let us abbreviate their conjunction by $Diag_\sigma(a,b)$. If b was a theorem in **CC**, then this diagonal formula is a theorem, too.

Let us now assume that $Diag_\sigma(a,b)$ is a theorem. Then each conjunct is a theorem, too, so they are true according our truth definition. The third conjunct says that the calculus with the code σ derives the string with the code b, i.e., B is a theorem of **CC**.

Now we have proven

Lemma 3. B is a theorem of **CC** iff $Diag_\sigma(a,b)$ is a theorem.

The Gödel sentence and its unprovability

Be A_0 the following formula with the code a_0:

$$\forall \mathfrak{x}_1 \neg Diag_\sigma(\mathfrak{x}, \mathfrak{x}_1).$$

The Gödel sentence and its unprovability

Be A_0 the following formula with the code a_0:

$$\forall \mathfrak{x}_1 \neg Diag_\sigma(\mathfrak{x}, \mathfrak{x}_1).$$

Let us diagonalize it and call the diagonalized formula G with the code g:

$$G = \forall \mathfrak{x}_1 \neg Diag_\sigma(a_0, \mathfrak{x}_1).$$

Be A_0 the following formula with the code a_0:

$$\forall \mathfrak{x}_1 \neg Diag_\sigma(\mathfrak{x}, \mathfrak{x}_1).$$

Let us diagonalize it and call the diagonalized formula G with the code g:

$$G = \forall \mathfrak{x}_1 \neg Diag_\sigma(a_0, \mathfrak{x}_1).$$

According to Lemma 3., G is a theorem of **CC** iff $Diag_\sigma(a_0, g)$ is a theorem.

Be A_0 the following formula with the code a_0:

$$\forall \mathfrak{x}_1 \neg Diag_\sigma(\mathfrak{x}, \mathfrak{x}_1).$$

Let us diagonalize it and call the diagonalized formula G with the code g:

$$G = \forall \mathfrak{x}_1 \neg Diag_\sigma(a_0, \mathfrak{x}_1).$$

According to Lemma 3., G is a theorem of **CC** iff $Diag_\sigma(a_0, g)$ is a theorem.

But from G follows $\neg Diag_\sigma(a_0, g)$. Therefore, if G is a theorem, then **CC** is inconsistent. Hence, G is not a theorem.

The truth of the Gödel sentence

The truth of the Gödel sentence

Suppose that G is false. Then there is a b_0 such that the closed atomic formula $Diag_\sigma(a_0, b_0)$ is true and hence provable.

The truth of the Gödel sentence

Suppose that G is false. Then there is a b_0 such that the closed atomic formula $Diag_\sigma(a_0, b_0)$ is true and hence provable.

Therefore the conjuncts
$D(\sigma)(\mathbf{F}'a_0)$, $D(\sigma)(b_0\mathbf{S}'a_0\mathbf{S}'[a_0]'\mathbf{S}'x')$, $D(\sigma)(b_0)$ are all true.

The truth of the Gödel sentence

Suppose that G is false. Then there is a b_0 such that the closed atomic formula $Diag_\sigma(a_0, b_0)$ is true and hence provable.

Therefore the conjuncts
$D(\sigma)(\mathbf{F}'a_0)$, $D(\sigma)(b_0\mathbf{S}'a_0\mathbf{S}'[a_0]'\mathbf{S}'x')$, $D(\sigma)(b_0)$ are all true.

From the second conjunct follows that b_0 cannot be different from g because the result of substituting the code a_0 into the formula with the code a_0 is the formula with the code g.

The truth of the Gödel sentence

Suppose that G is false. Then there is a b_0 such that the closed atomic formula $Diag_\sigma(a_0, b_0)$ is true and hence provable.

Therefore the conjuncts
$D(\sigma)(\mathbf{F}'a_0)$, $D(\sigma)(b_0\mathbf{S}'a_0\mathbf{S}'[a_0]'\mathbf{S}'x')$, $D(\sigma)(b_0)$ are all true.

From the second conjunct follows that b_0 cannot be different from g because the result of substituting the code a_0 into the formula with the code a_0 is the formula with the code g.

Therefore, $D(\sigma)(g)$ is true. But it means that the formula with the code g – i.e., G itself – is derivable in the calculus σ.

The truth of the Gödel sentence

Suppose that G is false. Then there is a b_0 such that the closed atomic formula $Diag_\sigma(a_0, b_0)$ is true and hence provable.

Therefore the conjuncts
$D(\sigma)(\mathbf{F}'a_0)$, $D(\sigma)(b_0\mathbf{S}'a_0\mathbf{S}'[a_0]'\mathbf{S}'x')$, $D(\sigma)(b_0)$ are all true.

From the second conjunct follows that b_0 cannot be different from g because the result of substituting the code a_0 into the formula with the code a_0 is the formula with the code g.

Therefore, $D(\sigma)(g)$ is true. But it means that the formula with the code g – i.e., G itself – is derivable in the calculus σ.

To sum up: G is not a theorem, but if it were false, then it would be provable. Therefore, it is a true but unprovable sentence.

Generalization

Generalization

Theorem: Be T a first-order theory such that

i. all the theorems of **CC** are provable in T;
ii. the class of the theorems of T is definable by some canonical calculus K;
iii. no false formula of **CC** is provable in T.

Then T is incomplete. There is a sentence in the language of T which is true but not provable.

Generalization

Theorem: Be T a first-order theory such that

i. all the theorems of **CC** are provable in T;

ii. the class of the theorems of T is definable by some canonical calculus K;

iii. no false formula of **CC** is provable in T.

Then T is incomplete. There is a sentence in the language of T which is true but not provable.

Be $K' = k$. If K derives a string f, then $D(k)(f')$ is provable in T (because it is provable in **CC**). So we have an analogue of Lemma 2. Then we can introduce $Diag_k(a/x, b)$ exactly as we have introduced $Diag_\sigma$. We can prove Lemma 3. for theorems of T instead of **CC**, and produce a Gödel sentence for T.

$Cons_\sigma$ and Step 1. to its unprovability

$Cons_\sigma$ and Step 1. to its unprovability

A definition:

$Th_\sigma(a) =_{df} D(\sigma)(\mathbf{F}'a) \wedge D(\sigma)(a)$

$Cons_\sigma$ and Step 1. to its unprovability

A definition:

$Th_\sigma(a) =_{df} D(\sigma)(\mathbf{F}'a) \wedge D(\sigma)(a)$

The $\mathcal{L}^{01}$-sentence expressing the consistency of **CC**:

$Cons_\sigma = \exists\mathfrak{x}(D(\sigma)(\mathbf{F}'\mathfrak{x}) \wedge \neg D(\sigma)(\mathfrak{x}))$

$Cons_\sigma$ and Step 1. to its unprovability

A definition:

$Th_\sigma(a) =_{df} D(\sigma)(\mathbf{F}'a) \wedge D(\sigma)(a)$

The $\mathcal{L}^{01}$-sentence expressing the consistency of **CC**:

$Cons_\sigma = \exists\mathfrak{x}(D(\sigma)(\mathbf{F}'\mathfrak{x}) \wedge \neg D(\sigma)(\mathfrak{x}))$

It was argued (in the metalanguage) that if the Gödel-sentence $B_0 (= G)$ is false, then it is a theorem of **CC**. This claim can be expressed in the language of **CC** as $\neg B_0 \supset Th(b_0)$.

$Cons_\sigma$ and Step 1. to its unprovability

A definition:
$Th_\sigma(a) =_{df} D(\sigma)(\mathbf{F}'a) \wedge D(\sigma)(a)$

The $\mathcal{L}^{01}$-sentence expressing the consistency of **CC**:
$Cons_\sigma = \exists\mathfrak{x}(D(\sigma)(\mathbf{F}'\mathfrak{x}) \wedge \neg D(\sigma)(\mathfrak{x}))$

It was argued (in the metalanguage) that if the Gödel-sentence $B_0(= G)$ is false, then it is a theorem of **CC**. This claim can be expressed in the language of **CC** as $\neg B_0 \supset Th(b_0)$.

But the argument for it can be reproduced within **CC**, too, so (Step 1.) $\neg B_0 \supset Th(b_0)$ is provable.
The axiom SUD (Substitution Uniquely Determined) is needed to show it:

$$\forall\mathfrak{x}_1\forall\mathfrak{x}_2\forall\mathfrak{x}_3\forall\mathfrak{x}_4$$
$$(D(\sigma)(\mathfrak{x}_3\mathbf{S}'\mathfrak{x}_2\mathbf{S}'\mathfrak{x}_1\mathbf{S}'\mathfrak{x}) \supset D(\sigma)(\mathfrak{x}_4\mathbf{S}'\mathfrak{x}_2\mathbf{S}'\mathfrak{x}_1\mathbf{S}'\mathfrak{x}) \supset \mathfrak{x}_3 = \mathfrak{x}_4)$$

Steps 2. and 3.

Be $C_0 = Diag_\sigma(a_0, b_0)$ with the code c_0.

Steps 2. and 3.

Be $C_0 = Diag_\sigma(a_0, b_0)$ with the code c_0.
We know that $\Gamma_0 \vdash B_0$ iff $\Gamma_0 \vdash C_0$. This biconditional can be proven within **CC** again, i.e. $\Gamma_0 \vdash B_0 \leftrightarrow C_0$.

Steps 2. and 3.

Be $C_0 = Diag_\sigma(a_0, b_0)$ with the code c_0.
We know that $\Gamma_0 \vdash B_0$ iff $\Gamma_0 \vdash C_0$. This biconditional can be proven within **CC** again, i.e. $\Gamma_0 \vdash B_0 \leftrightarrow C_0$.

By the definition of Th_σ, it follows that $\Gamma_0 \vdash Th(b_0) \supset Th(c_0)$.

Steps 2. and 3.

Be $C_0 = Diag_\sigma(a_0, b_0)$ with the code c_0.
We know that $\Gamma_0 \vdash B_0$ iff $\Gamma_0 \vdash C_0$. This biconditional can be proven within **CC** again, i.e. $\Gamma_0 \vdash B_0 \leftrightarrow C_0$.

By the definition of Th_σ, it follows that $\Gamma_0 \vdash Th(b_0) \supset Th(c_0)$.

Using the result of Step 1., we get
(Step 2.) $\Gamma_0 \vdash \neg B_0 \supset Th(c_0)$

Steps 2. and 3.

Be $C_0 = Diag_\sigma(a_0, b_0)$ with the code c_0.
We know that $\Gamma_0 \vdash B_0$ iff $\Gamma_0 \vdash C_0$. This biconditional can be proven within **CC** again, i.e. $\Gamma_0 \vdash B_0 \leftrightarrow C_0$.

By the definition of Th_σ, it follows that $\Gamma_0 \vdash Th(b_0) \supset Th(c_0)$.

Using the result of Step 1., we get
(Step 2.) $\Gamma_0 \vdash \neg B_0 \supset Th(c_0)$

We know that if $\Gamma_0 \vdash C_0$, then $\Gamma_0 \vdash B_0$, and if $\Gamma_0 \vdash B_0$, then $\Gamma_0 \vdash \neg C_0$.

Steps 2. and 3.

Be $C_0 = Diag_\sigma(a_0, b_0)$ with the code c_0.
We know that $\Gamma_0 \vdash B_0$ iff $\Gamma_0 \vdash C_0$. This biconditional can be proven within **CC** again, i.e. $\Gamma_0 \vdash B_0 \leftrightarrow C_0$.

By the definition of Th_σ, it follows that $\Gamma_0 \vdash Th(b_0) \supset Th(c_0)$.

Using the result of Step 1., we get
(Step 2.) $\Gamma_0 \vdash \neg B_0 \supset Th(c_0)$

We know that if $\Gamma_0 \vdash C_0$, then $\Gamma_0 \vdash B_0$, and if $\Gamma_0 \vdash B_0$, then $\Gamma_0 \vdash \neg C_0$.

It follows that $\Gamma \vdash Th(c_0) \supset Th(\neg' c_0)$.

Steps 2. and 3.

Be $C_0 = Diag_\sigma(a_0, b_0)$ with the code c_0.
We know that $\Gamma_0 \vdash B_0$ iff $\Gamma_0 \vdash C_0$. This biconditional can be proven within **CC** again, i.e. $\Gamma_0 \vdash B_0 \leftrightarrow C_0$.

By the definition of Th_σ, it follows that $\Gamma_0 \vdash Th(b_0) \supset Th(c_0)$.

Using the result of Step 1., we get
(Step 2.) $\Gamma_0 \vdash \neg B_0 \supset Th(c_0)$

We know that if $\Gamma_0 \vdash C_0$, then $\Gamma_0 \vdash B_0$, and if $\Gamma_0 \vdash B_0$, then $\Gamma_0 \vdash \neg C_0$.

It follows that $\Gamma \vdash Th(c_0) \supset Th(\neg' c_0)$.

Therefore, using Step 2. and propositional logic:
Step 3. $\Gamma_0 \vdash \neg B_0 \supset (Th(c_0) \wedge Th(\neg' c_0))$

Step 4. and the finish of the proof

Step 4. and the finish of the proof

$Th(c_0) \wedge Th(\neg' c_0)$ is the encoded form of a contradiction. For any sentences A, B, $(A \wedge \neg A) \supset B$ is a provable formula of propositional logic and it can be used for this encoded form, too. Therefore,

Step 4. $\Gamma_0 \vdash (Th(c_0) \wedge Th(\neg' c_0)) \supset \neg Cons_\sigma$.

Step 4. and the finish of the proof

$Th(c_0) \wedge Th(\neg' c_0)$ is the encoded form of a contradiction. For any sentences A, B, $(A \wedge \neg A) \supset B$ is a provable formula of propositional logic and it can be used for this encoded form, too. Therefore,

Step 4. $\Gamma_0 \vdash (Th(c_0) \wedge Th(\neg' c_0)) \supset \neg Cons_\sigma$.

From Step 3. and Step 4. it follows that

$\Gamma_0 \vdash \neg B_0 \supset \neg Cons_\sigma$.

By propositional logic,

$\Gamma_0 \vdash Cons_\sigma \supset B_0$.

Step 4. and the finish of the proof

$Th(c_0) \wedge Th(\neg' c_0)$ is the encoded form of a contradiction. For any sentences A, B, $(A \wedge \neg A) \supset B$ is a provable formula of propositional logic and it can be used for this encoded form, too. Therefore,

Step 4. $\Gamma_0 \vdash (Th(c_0) \wedge Th(\neg' c_0)) \supset \neg Cons_\sigma$.

From Step 3. and Step 4. it follows that

$\Gamma_0 \vdash \neg B_0 \supset \neg Cons_\sigma$.

By propositional logic,

$\Gamma_0 \vdash Cons_\sigma \supset B_0$.

Therefore, if $Cons_\sigma$ were provable, then B_0, the Gödel sentence would be provable, too. But from the first incompleteness theorem we know that the Gödel sentence is not provable, and therefore $Cons_\sigma$ can't be provable, either. Q.e.d.