The first-order theory of canonical calculi

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 - $\forall \mathfrak{x}_1 \exists \mathfrak{x}(\mathfrak{x}_1 \neq \vartheta \supset (\mathfrak{x}_1 = \mathfrak{x}\alpha \vee \mathfrak{x}_1 = \mathfrak{x}\beta \vee \mathfrak{x}_1 = \mathfrak{x}\xi \vee \mathfrak{x}_1 = \mathfrak{x} \gg \\ \vee \mathfrak{x}_1 = \mathfrak{x}^*))$

Remark: The textbook defines the theorems of \mathbb{CC}^* by a canonical calculus Σ^* . We omit this step; but you can find the axioms of this slide as rules 61-80. of Σ^* on p. 80. of the textbook. The notation is a little bit different.



To obtain the axioms about calculi, we can simply translate the 34 rules of the hypercalculus \mathbf{H}_3 into \mathcal{L}^{1*} -propositions. The rules of the translation are the following:

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- Calculus variables x, y, z, ... are substituted by the \mathcal{L}^{1*} -variables $\mathfrak{r}, \mathfrak{r}_1, ...$



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The axioms of \mathbb{CC}^* are the 20 language radix-axioms plus the 34 axioms obtained from the rules of \mathbf{H}_3 . E.g., the rules 12. and 13. of \mathbf{H}_3 (defining the extension of K) become the following axioms:

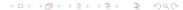
- $\bullet \ \forall \mathfrak{x}(R(\mathfrak{x}) \supset K(\mathfrak{x}))$
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The above rules of translation apply to any string derivable in \mathbf{H}_3 . Let us denote the translation on the string f into a \mathcal{L}^{1*} -formula Tr(f).



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- Closed atomic formulas containing the predicates I, L, W, V, T, R, K, F, S are true iff they are true according to the intended interpretation. I.e., $\lceil I(s) \rceil$ is true iff the string s is an index, $\lceil K(s) \rceil$ is true iff s is a code of a calculus, $\lceil S(s)(t)(v)(u) \rceil$ is true iff by substituting the word (variable-free string) v for the variable u in the string t, we get s, etc.

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These two stipulations are effective, so the reference to the intended interpretation is not problematic.

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Theorem: If $\mathbf{H}_3 \mapsto f$, then Tr(f) is provable in \mathbf{CC}^* .

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Suppose we have an algorithm to decide which sentences of \mathcal{L}^{1*} are theorems of \mathbf{CC}^* . In this case, we could decide which sentences of the form A(c) (where c is a numeral) are theorems. But this would mean that we could decide which numerals are autonomous - in contradiction to our earlier result that the class of autonomous numerals is not decidable.

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Theorem(Church-Turing-Markov): First-order logic is not decidable.

I. e., there is no algorithm for every first-order language that decides about every formula whether it is a logical truth (consequence of the empty set of formulas) or not.

E.g., for \mathcal{L}^{1*} there is no such algorithm. Because otherwise we had an algorithm to decide which formulas of the form

 $\mathbf{A}\mathbf{x} \supset A(c)$ are logical truths (where $\mathbf{A}\mathbf{x}$ is the conjunction of all axioms of $\mathbf{C}\mathbf{C}^*$ and c is a numeral). This would imply the decidability of the class of autonomous numerals again.

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The interesting case is when a theory is incomplete because it is too strong, and therefore the incompleteness cannot be cured by extending the theory.

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The class of axioms Γ_0 of **CC** comes from the axioms of **CC*** by omitting the last nine axioms corresponding the rules 26.-34. of \mathbf{H}_3 (i.e, it contains the axioms that translate the rules of \mathbf{H}_2 but not the further rules of \mathbf{H}_3 governing the predicates omitted) and by adding SUD.

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We can apply the truth definition we have specified at the last class. We will show that SUD is true according to this definition, too. Therefore, the theorems of \mathbf{CC} are all true and the theory is consistent.