Intuitionism – and a little overview

Historical introduction to the philosophy of mathematics

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Intuitionism:
Create a new mathematics based on intuitively clear notions and stricter forms of argumentation.

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Doubts about the law of excluded middle (LEM).

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– Objects whose existence can be proved only indirectly (e.g. non-Lebesgue-measurable sets).
– “Funny” functions. Real analysis as an art of producing counterexamples.

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Anti-realist stance: truth is not independent of our knowledge. It is created by us (at least in some respect). Julius König: A logic with LEM is the logic of God because he is omniscient – it is not *our* logic.

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Michael Dummett (1925-2011): Abandons psychologism of Brouwer but keeps anti-realism. You understand the meaning of a mathematical proposition if you are able to recognize whether a construction is a proof of the proposition or not.

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Brouwer: mathematical objects are created in the mind, by the two *acts of intuitionism*.

The first act

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"Completely separating mathematics from mathematical language and hence from the phenomena of language described by theoretical logic, recognizing that intuitionistic mathematics is an essentially languageless activity of the mind having its origin in the perception of a move of time. This perception of a move of time may be described as the falling apart of a life moment into two distinct things, one of which gives way to the other, but is retained by memory. If the twoity thus born is divested of all quality, it passes into the empty form of the common substratum of all twoities. And it is . . . this empty form, which is the fundamental intuition of mathematics."

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"[The] intuition of two-oneness creates not only the numbers one and two, but also all finite ordinal numbers, inasmuch as one of the elements of the two-oneness may be thought of as a new two-oneness, which process may be repeated indefinitely; this gives rise still further to the smallest infinite ordinal number ω."

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"Admitting two ways of creating new mathematical entities: firstly in the shape of more or less freely proceeding infinite sequences of mathematical entities previously acquired . . . ; secondly in the shape of mathematical species, i.e. properties supposable for mathematical entities previously acquired, satisfying the condition that if they hold for a certain mathematical entity, they also hold for all mathematical entities which have been defined to be 'equal' to it . . . ."

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The function of the second act is that it allows to create free choice sequences of previously established mathematical objects. This is the basis of the intuitionist continuum.

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"For this reason the intuitionist can never feel assured of the exactness of a mathematical theory by such guarantees as the proof of its being noncontradictory, the possibility of defining its concepts by a finite number of words. "

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Vienna Circle: linguistic convention instead of timeless objects. It makes Ramsey's justifications of the axioms of choice and infinity inaccessible.

Formalism: mathematics has no special object, it has a special method (formal deduction). Applicability and application is a question outside mathematics, therefore the metaphysical nature of the objects of our theories is irrelevant for mathematics.
"Existence in mathematics is nothing but consistency" (Hilbert); consistent (first-order) theories do have models and that's all we need.

Brouwer's example of the opposition between the intuitionist and the formalist

"Let us now consider the concept: 'denumerably infinite ordinal number.' From the fact that this concept has a clear and well-defined meaning for both formalist and intuitionist, the former infers the right to create the 'set of all denumerably infinite ordinal numbers', the power of which he calls $\aleph_1$, a right not recognized by the intuitionist. Because it is possible to argue to the satisfaction of both formalist and intuitionist, first, that denumerably infinite sets of denumerably infinite ordinal numbers can be built up in various ways, and second, that for every such set it is possible to assign a denumerably infinite ordinal number, not belonging to this set, the formalist concludes: '$\aleph_1 > \aleph_0$ ',a proposition that has no meaning for the intuitionist."

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Brouwer about logic: "The ... point of view that there are no non-experienced truths and that logic is not an absolutely reliable instrument to discover truths has found acceptance with regard to mathematics much later than with regard to practical life and to science."

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- Proof of $A \to B$ is a construction which transforms any proof of A into a proof of B.

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- Proof of $A \rightarrow B$ is a construction which transforms any proof of A into a proof of B.
- Proof of $\exists x A(x)$: presenting a member d of the domain and proving $A(d)$.

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- Proof of $\neg A$: proof of $A \rightarrow \perp$.

This is not a (formal) definition because it is based on an informal notion of construction.