Ramsey's Logicism continued

Historical introduction to the philosophy of mathematics

András Máté

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Ramseyian logicism: mathematics should consist of tautologies.

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Semantical paradoxes involve the notion of 'meaning' (denoting) and they are irrelevant for mathematics. Logical paradoxes involve only logical and mathematical concepts and show that something went wrong in our logic (mathematics).

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Unquestionably correct and sufficient to remove the logical paradoxes.

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But it blocks important mathematical ideas and arguments (math. induction, Dedekind cut). That is why the axiom of reducibility is needed for Russell.

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E. g.

$$F(a), \quad \exists x(F(x) \wedge x = a)$$

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Ramsey: this is the root of the whole problem of reducibility.

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A function is predicative$_{Ramsey}$ iff it is the truth function of arguments that are either atomic functions of individuals or propositions.

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Ramsey: yes, but this sort of circularity is not vicious. A value of F for an individual a, the proposition Fa is the conjunction of all the propositions of the form ϕa - including Fa itself. No reference to a class of which F is a member, but to the members of that class only.

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5. $H(\text{'heterological'}) \Leftrightarrow \neg H(\text{'heterological'})$

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Such cases of denoting are more difficult and the relation R is not legitimately extended to them.

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E. g. in the paradox of the least natural number not nameable in fewer than nineteen syllables we create a new definition that is of higher order than the definitions it refers to.

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Choice: in the framework of the *Principia* it is empirical. But in Ramsey's interpretation, it becomes a tautology. But not necessarily provable.

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Therefore, the Axiom of Infinity is, if it is true, a tautology, but can't be proved. It must be postulated.