A budget of paradoxes

Historical introduction to the philosophy of mathematics

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We have proved a logical falsity from the (unlimited) comprehension using only logical rules.

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Suppose (for contradiction) that $H_0 = f(h)$.

$$h \in f(h) \leftrightarrow h \notin f(h)$$

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Because of the first conjunct in the scope of $\exists$, any concept F which makes the existential quantification true is true for just the same objects as R (because of Axiom V). Therefore, the right side is true iff $\neg R({}^{\vee}R)$.

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Let me introduce a collection of relevant paradoxes. (*A budget of paradoxes*: De Morgan 1872.)

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$L \leftrightarrow \neg L$

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It is smaller than its successor.

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Known as Grelling-Nelson, Weyl, or simply heterological-paradox.

Two more famous paradoxes

The smallest number not definable in English by 72 characters

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Let us enumerate all these numbers in the sequence a_k. Consider the following real number $a = 0.d_1 d_2 ... d_n ...$:
$d_n = 6$ if the nth digit after the decimal point of of a_n is 5 and $d = 5$ otherwise.

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a differs from any member of our sequence, but it is defined.

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The students write the test on Wednesday and they get really surprised.

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G is an ordinary game between two players iff it finishes in finitely many steps. H is the following hypergame: the first player chooses an ordinary game, and then they play it. Is H an ordinary game or not?

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Russell's principle forbids self-reference. It is apparently enough to avoid the previous paradoxes.

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It is a liar-like, but infinitary paradox that does not violate the vicious circle principle and does not contain any sort of self-reference.

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1. Improve logic and produce a unique general theory free of risks (logicism)
2. Risky theories but a reliable metatheory (formalism)
3. Abandon the priority of logic in favor of a more reliable basis (intuitionism)