Dedekind's numbers

András Máté

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A definition of abstract objects introduced by an abstraction principle is consistent *relative to set theory* if the equivalence classes generated by the principle are sets.

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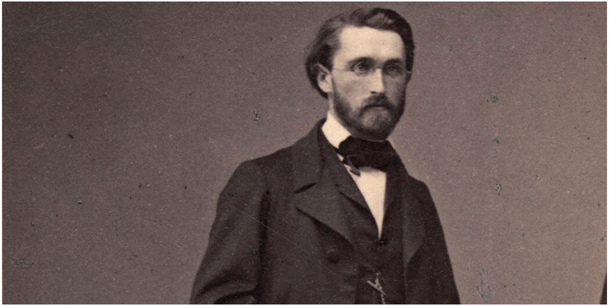

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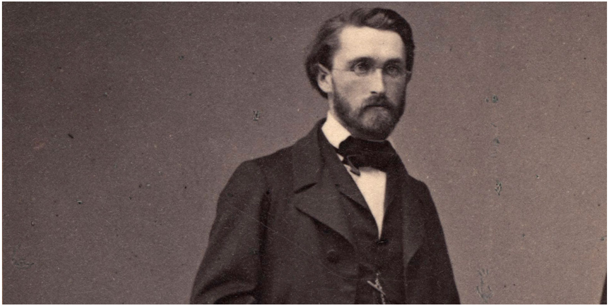

The grandfather of mathematical structuralism

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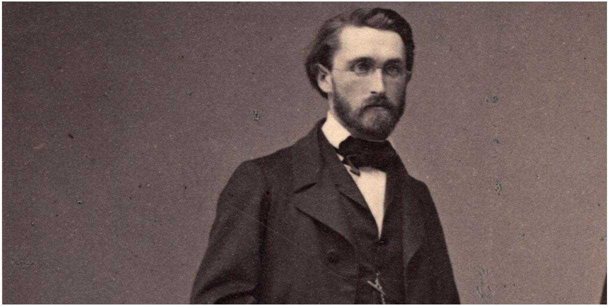

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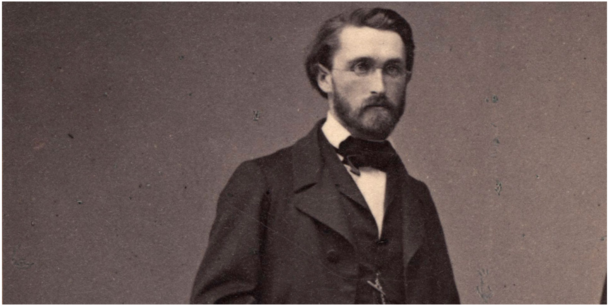

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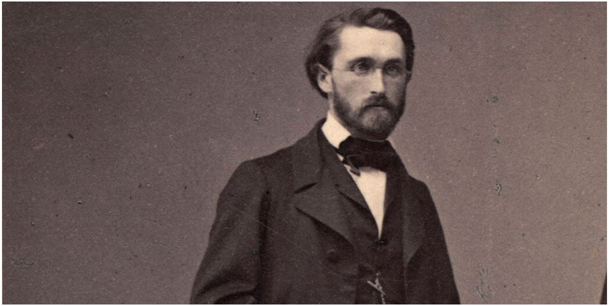

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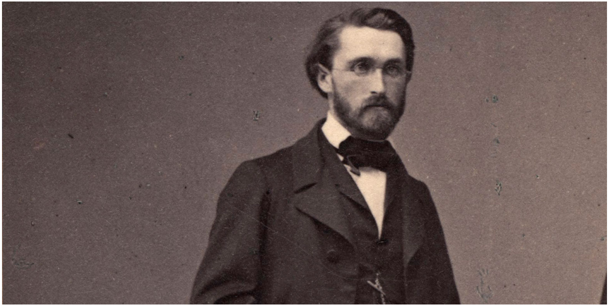

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1. Bourbaki circle from the 1930's
2. Paul Benacerraf: „What numbers could not be" (1965)
3. William Lawvere's works on category theory (from the 1960's)

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Dedekind cut: Divide the rational numbers into two classes so that every member of the first (lower) class is less than any member of the second (upper) class. Such a classification is called cut.

There are three sorts of cuts:

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But what are the natural numbers?

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[A function φ is injective iff $\varphi(x) = \varphi(y) \to x = y$]

$S' = \varphi(S)$ is the system consisting of the φ-pictures of the members of S. If φ is a similarity transformation, then it has a converse that is a similarity transformation again and φ is an one-to-one correspondence between the members of S and S'.

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Theorem of complete induction: For any systems Σ and $A \subseteq \Sigma$, if for any $x \in A_0 \cap \Sigma$, $\varphi(x) \in A_0 \cap \Sigma$, then $A_0 \subseteq \Sigma$.

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66. Theorem. There exist infinite systems.

Proof.* My own realm of thoughts, i. e., the totality S of all things, which can be objects of my thought, is infinite. For if s signifies an element of S, then is the thought s', that s can be object of my thought, itself an element of S. If we regard this as transform $\phi(s)$ of the element s then has the transformation ϕ of S, thus determined, the property that the transform S' is part of S; and S' is certainly proper part of S, because there are elements in S (e. g., my own ego) which are different from such thought s' and therefore are not contained in S'. Finally it is clear that if a, b are different elements of S, their transforms a', b' are also different, that therefore the transformation ϕ is a distinct (similar) transformation (26). Hence S is infinite, which was to be proved.

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Natural numbers: the elements of any simply infinite system N if
we entirely neglect the special character of the elements; simply retaining their distinguishability and taking into account only the relations to one another in which they are placed by the order-setting transformation ϕ

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To sum up, the axioms of second-order PA hold for simply infinite systems.

In other words, simply infinite systems are models of second order Peano arithmetics. The converse is also true: every model of second-order PA is a simply infinite system.

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Therefore, second-order Peano arithmetics (the set of *semantical* consequences of second-order Peano axioms) is negation complete.

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Second-order logic cannot have a semantically complete calculus. Because if it had, then we could derive every semantical consequence from the second-order Peano axioms an we got a negation complete axiomatic extension of first-order Peano arithmetics.

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What is arithmetical truth? A simple-looking answer: a theorem of second-order PA. But the appearance of simplicity is misleading here.