A historical introduction to the philosophy of mathematics

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This picture about mathematics may be called the dogmatic view.

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Figures or *calculi* (little stones used to exemplify numbers) are just auxiliary tools to help our reason.

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Mathematics is nothing but further developed logic.

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The fundamental truths are based on the properties of human cognitive capacity.

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Bolzano's remark: it should be proven that there is a number which is the sum of the series.

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Bolzano: the paradoxes of infinite(ly small or large) quantities can be resolved by scrupulous (re-)defining of the basic concepts and proving everything that seems to be obvious.

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Bolzano on infinite manifolds

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His conclusion: The paradoxes of infinite pluralities can't be resolved at all.

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Theory of infinite sets, infinite cardinals and ordinals.

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Is the Cantorian concept of set as clear and well supported as it seems to be?

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18th century: many attempts to prove the axiom (typically on the indirect way).

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Some twenty years later people prove the equiconsistency of Euclidean and Bolyai-Lobachevsky-geometry (Cayley-Klein model). Therefore, there is no way to decide which one is the true science of the space.

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Problems in geometry B.: Hidden axioms in Euclid

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Another remaining problem.