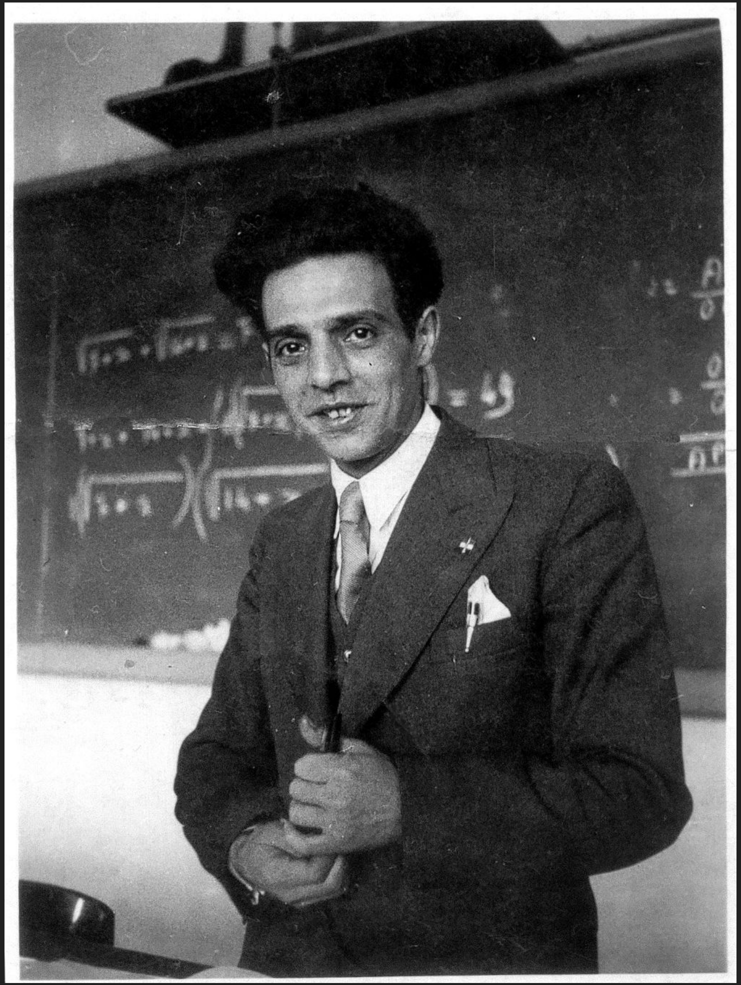
Peano Axioms

Lars Stepan Laichter

GIUSEPPE PEANO

- *1858 – 1932*
- Italian
- Peano axioms
  published in 1889
  as *The principles of arithmetic presented by a new method*



SOURCE: http://www.peano2008.unito.it/scuola/foto/gliozzi.jpg

What are the Peano Axioms?

- Expresses logical rules and operations of natural numbers
- Defined by 9 axioms
- Recursive definition

9 Axioms

1. ...
2. Equality is reflexive
3. Equality is symmetric
4. Equality is transitive
5. Natural numbers are closed under equality
6. ...
7. ...
8. ...
9. ...

5 Axioms

1. Zero is a natural number
2. The successor of any natural number is a natural number
3. Zero has no predecessor
4. Different natural numbers have different successors
5. If zero has a property and if any natural number has that property, then its successor must have that property, then all natural numbers must have that property

Axioms of addition

1. Any number plus zero is that number
2. Any number plus the successor of any other number equals the successor of the sum of the two numbers

Axioms of multiplication

1. Any number times zero is zero
2. Any number times the successor of any other number equals the product of the two numbers plus the first number

Symbols

- ‘o’ zero
- ‘S’ successorship
- ‘+’ addition
- ‘·’ multiplication
- ‘n’ and ‘m’ for variables
- ‘$\supset$’ implies

Peano's notation

$$ab.cd{:}ef.gh \;.\!\overset{\cdot}{}\!. \; k$$

$$(((ab)(cd))((ef)(gh)))k.$$

Successorship

$0 = \text{Zero}$

$S(0) = \text{One}$

$S(S(0)) = \text{Two}$

$S(S(S(0))) = \text{Three}$

$S(n) = \text{The number after } n$

$S(S(m)) = \text{The number two after } m$

Operations

$$S(0) + S(S(0)) = S(S(S(0)))$$

$$0 \cdot S(S(0)) = 0$$

Axioms

1. PN1

$$(\exists n)(o = n)$$

2. PN2

$$(\square n)(\exists m)(m = S(n))$$

3. PN3

$$\sim(\exists n)(S(n) = 0)$$

4. PN4

$$(\square n)(\square m)((S(m) = S(n)) \supset (n=m))$$

Can this go both ways?

$$(\square n)(\square m)((n=m) \supset (S(m) = S(n)))$$

Can this go both ways?

$$(\square n)(\square m)((n=m) \supset (S(m) = S(n)))$$

1.	$n=m$	(ACP)
2.	$a=S(n)$	(PN2, EI, UI)
3.	$a=S(m)$	(1., 2., Id.)
4.	$S(n)=S(m)$	(2., 3., Id.)
5.	$(n=m) \supset (S(m) = S(n))$	(1.–4. CP)
6.	$(\square n)(\square m)((n=m) \equiv (S(m) = S(n)))$	(5., UG)

5. PN5

$$(\square A)((A(0) \;\&\; (\square n)(A(n) \supset A(S(n))) \supset (\square m)(A(m)))$$

Operations – *Addition*

$$(\square n)((o+n)=n)$$

$$(\square n)(\square m)((n+S(m))=S(n+m))$$

Addition example

$$n + 3 = n + S(2)$$
$$= S(n+2)$$
$$= S(S(S(n)))$$

Operations – *Multiplication*

$$(\square n)((n \cdot 0)=0)$$

$$(\square n)(\square m)((n \cdot S(m))=((n \cdot m)+n)$$

Thank you!