Gödel's First Incompleteness Theorem
Original form

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Theorem: Be $A(v_1, v_2)$ a formula that enumerates P^* in $\mathcal{S}$, a the Gödel number of $\forall v_2 \neg A(v_1, v_2)$ and G the sentence $\forall v_2 \neg A(\bar{a}, v_2)$. Then:

1. if $\mathcal{S}$ is (simply) consistent, then G is not provable;
2. if $\mathcal{S}$ is ω-consistent, then G is not refutable, either.

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By assumption, $\mathcal{S}$ is axiomatizable, i.e. P is Σ_1. We proved that the adjoint set of any Σ_1 set is Σ_1, too. Hence P^* is Σ_1. By assumption, P^* is enumerable and according to the previous propositions, $\mathcal{S}$ is incomplete.

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$\mathbf{A}_2$ If every true Σ_0 sentence is provable in $\mathcal{S}$, then every Σ_1 set and relation is enumerable.

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$\mathbf{A}_2$ If every true Σ_0 sentence is provable in $\mathcal{S}$, then every Σ_1 set and relation is enumerable.

If $R(v_1, \ldots, v_n)$ is a Σ_1 relation, then there is an $S(v_1, \ldots, v_n, v_{n+1})$ Σ_0 relation s.t.

$$R(v_1, \ldots, v_n) \leftrightarrow \exists y S(v_1, \ldots, v_n, y)$$

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If $R(k_1, \ldots, k_n)$ holds, then for some k, $S(k_1, \ldots, k_n, k)$ holds and therefore the Σ_0 sentence $F(\bar{k}_1, \ldots . \bar{k}_n, \bar{k})$ is true. By assumption, it is provable.

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If $R(k_1, \ldots, k_n)$ does not hold, then for no k holds $S(k_1, \ldots, k_n, k)$. Therefore for any k, the sentence $F(\bar{k}_1, \ldots . \bar{k}_n, \bar{k})$ is false. Its negation is true and Σ_0, therefore provable, and the sentence itself is refutable.

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From $\mathbf{A}_1$ and $\mathbf{A}_2$ it follows

A. If $\mathcal{S}$ is axiomatizable, ω-consistent and every true Σ_0 sentence is provable in $\mathcal{S}$, then $\mathcal{S}$ is incomplete.

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$\mathbf{A}^*$: If $\mathcal{S}$ is axiomatizable, ω-consistent and no false Σ_0 sentence is provable in $\mathcal{S}$, then $\mathcal{S}$ is incomplete.

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If every true Σ_0 sentence is provable, then $\mathcal{S}$ is incomplete by A. If not, then there is a true Σ_0 sentence A that is not provable, and $\neg A$ is not provable, either, because it is false.

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Another proof for A*:
Be $\mathcal{S}$ axiomatizable, $R(x, y)$ an arbitrary Σ_0 relation with the domain P^*, $A(v_1, v_2)$ the Σ_0 formula expressing it, a the Gödel number of the formula $\forall v_2 \neg A(v_1, v_2)$ and G the sentence $\forall v_2 \neg A(\bar{a}, v_2)$.

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Be $\mathcal{S}$ axiomatizable, $R(x, y)$ an arbitrary Σ_0 relation with the domain P^*, $A(v_1, v_2)$ the Σ_0 formula expressing it, a the Gödel number of the formula $\forall v_2 \neg A(v_1, v_2)$ and G the sentence $\forall v_2 \neg A(\bar{a}, v_2)$.

1. Suppose G is provable. Then $a \in P^*$, therefore there is an n s.t. $A(\bar{a}, \bar{n})$ is true (because a is in the domain of R). But G entails the sentence $\neg A(\bar{a}, \bar{n})$ that is a false Σ_0 sentence.

A^*, continuation

2. Suppose that G is refutable and $\mathcal{S}$ is ω-consistent. Now $\exists y A(\bar{a}, y)$ is provable. By ω-consistency, there is an n s.t. $\neg A(\bar{a}, \bar{n})$ is not provable. $\mathcal{S}$ is consistent, therefore G is not provable, $a \notin P^*$ and $A(\bar{a}, \bar{m})$ is false for any m. So $\neg A(\bar{a}, \bar{n})$ is a true but not provable Σ_0 sentence.

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3. Assume now that $\mathcal{S}$ is complete, consistent and no false Σ_0 sentence is provable. Then by 1., G is not provable. By completeness, it is refutable but every true Σ_0 sentence is provable. Therefore by 2., $\mathcal{S}$ is ω-inconsistent.

Homeworks

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1. Prove that if all true Σ_0 sentences are provable in $\mathcal{S}$, and $\mathcal{S}$ is ω-consistent, then all Σ_1 sets are representable.

2. Be $F(v_1, v_2)$ a formula that represents the same relation that it expresses. Suppose that for every m and n, $F(\bar{n}, \bar{m})$ is either provable or refutable, and $\mathcal{S}$ is ω-consistent. Prove that $\exists v_2 F(v_1, v_2)$ represents the same set that it expresses.

3. Prove the following dual of the second Theorem of the previous class:
   Suppose $B(v_1, v_2)$ a formula that enumerates R^* in $\mathcal{S}$, b the Gödel number of $\exists v_2 B(v_1, v_2)$ and G the sentence $\forall v_2 \neg B(\bar{b}, v_2)$. Then:
   1. if $\mathcal{S}$ is (simply) consistent, then G is not provable;
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We did prove **A** (and even the stronger theorem $\mathbf{A}^*$: If $\mathcal{S}$ is axiomatizable, ω-consistent and no false Σ_0 sentence is provable in $\mathcal{S}$, then $\mathcal{S}$ is incomplete.)

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Now we should prove **B.**

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Suppose

- C_1 Every atomic Σ_0-sentence is correctly decidable;
- C_2 If $F(w)$ is a Σ_0-formula with the only free variable w and $F(\bar{0}), \ldots, F(\bar{n})$ are all provable, then $(\forall w \leq n)F(w)$ is provable, too.

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Prove that every Σ_0-sentence is correctly decidable by induction on the degree of the sentence.

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- If P is $\bar{m} + \bar{n} = \bar{k}$, then for some $l \neq k$, $\bar{m} + \bar{n} = \bar{l}$ is true and by D_1, provable. By D_2, $\bar{k} \neq \bar{l}$ is provable, too, and they imply $\neg P$.

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- If P is $\bar{m} + \bar{n} = \bar{k}$, then for some $l \neq k$, $\bar{m} + \bar{n} = \bar{l}$ is true and by D_1, provable. By D_2, $\bar{k} \neq \bar{l}$ is provable, too, and they imply $\neg P$.
- Similarly for a P of the form $\bar{m} \cdot \bar{n} = \bar{k}$.

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- (R) Non-logical axioms are instances of the following 5 schemes:

  - Ω_1 $\bar{m} + \bar{n} = \bar{k}$, where $m + n = k$.
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(R_0) is a subsystem of (Q_0) and (R) is a subsystem of (Q). We need metalanguage induction to prove that the axioms of (R) are provable in (Q).

Step B. and Gödel's Theorem

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From A. and B.

There is a Σ_0 formula $A(v_1, v_2)$ which enumerates P^* in P.A.
$E_a = \forall v_2 \neg A(v_1, v_2)$ is a formula whose negation represents P^*.
$G = \forall v_2 \neg A(\bar{a}, v_2)$ is not provable in P.A. if P.A. is consistent and it is not refutable, either if P.A. is ω-consistent.