Abstract incompleteness theorems

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$\mathcal{S}$ is <u>ω-inconsistent</u> if for some formula $F(w)$, the sentence $\exists w F(w)$ is provable but the sentences $F(\bar{0})$, $F(\bar{1}), \dots F(\bar{n}), \dots$ are all refutable. <u>ω-consistent</u> in the other case.

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$\mathcal{S}$ is recursively axiomatizable if P is Σ_1. (Synonyms: (simply) axiomatizable, recursively enumerable, formal, Σ_1-system.)

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B. All true Σ_0-sentences are provable in P.A.

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$F(v_1,\ v_2, \ldots,\ v_n)$ represents the set of n-tuples A if $((k_1,\ k_2, \ldots,\ k_n) \in A$ iff $F(\bar{k}_1,\ \bar{k}_2, \ldots,\ \bar{k}_n)$ is provable).

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Let G be a true but not provable sentence of P.A. $G \wedge v_1 = v_1$ expresses the set of all numbers but represents the empty set.

An abstract incompleteness theorem

Let $\mathcal{S}$ be a system, P and R the set of Gödel numbers of provable resp. refutable sentences. P^* resp. R^* is the set of numbers n s.t. $E_n[\bar{n}]$ is provable resp. refutable.

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Hence, for any formula $H(v_1)$ with the Gödel number h,
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Theorem: Let $\mathcal{S}$ be consistent, $E_h = H(v_1)$ a formula whose negation represents P^* in $\mathcal{S}$. Then $H(\bar{h})$ is undecidable.

Since the negation of $H(v_1)$ represents P^*, for any n, $n \in P^*$ iff $H(\bar{n})$ is refutable in $\mathcal{S}$. Therefore, $H(\bar{h})$ is refutable iff $h \in P^*$ and (according to the previous claim) iff $h \in R^*$. I. e., $H(\bar{h})$ is either both provable and refutable or neither provable nor refutable. By consistency, the second.

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Proof 1: If $F(v_1)$ represents P^* in $\mathcal{S}$, then $\neg\neg F(v_1)$ represents P^*, too. $\neg F(v_1)$ is a formula whose negation represents P^*, so the conditions of the previous theorem are satisfied.

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Proof 2: Let $H(v_1)$ represent P^* in $\mathcal{S}$ and k the Gödel number of $\neg H(v_1)$. Then $H(\bar{k})$ is undecidable.

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Proof: Let $H(v_1)$ represent R^* and its Gödel number be h. Then $H(\bar{h})$ is provable iff $h \in R^*$ iff $H(\bar{h})$ is refutable.

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1. Finish Proof 2. Why is $H(\bar{k})$ undecidable?

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2. Prove that $\widetilde{P^*}$ (the complement set of P^*) is not representable (and this is independent of consistency).

Homeworks:

3. Suppose $\widetilde{P}^*$ is representable in the $\mathcal{S}'$ consistent extension of $\mathcal{S}$. Prove that $\mathcal{S}$ is incomplete.

Homeworks:

4. Be $\mathcal{S}$ a subsystem of $\mathcal{N}$ (i.e., all provable sentences be true), and h the Gödel number of $H(v_1)$.
   Suppose *the negation of* $H(v_1)$ both represents and expresses P^*. Then $H(\bar{h})$ is undecidable. But is it true or false?
   Suppose now $H(v_1)$ both represents and expresses R^*. Is in this case $H(\bar{h})$ true or false?

Enumerability

The formula $F(v_1, v_2)$ enumerates the set A in the system $\mathcal{S}$ if

1. (i) if $n \in A$, then there is an m s.t. $F(\bar{n}, \bar{m})$ is provable;
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More generally, $F(v_1, \ldots, v_n, v_{n+1})$ enumerates the relation $R(v_1, \ldots, v_n)$ in $\mathcal{S}$ if

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A set resp. a relation is enumerable if there is a function which enumerates it.

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Suppose $\exists v_2 F(\bar{n}, v_2)$ is provable. If n were not in A, then $F(\bar{n}, \bar{0}),\ F(\bar{n}, \bar{1}), \ldots,\ F(\bar{n}, \bar{m}), \ldots$ would be all refutable, and $\mathcal{S}$ would be ω-inconsistent.

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Consequence: If $\mathcal{S}$ is ω-consistent and either P^* or R^* is enumerable, then $\mathcal{S}$ is not complete.

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Theorem: Be $A(v_1, v_2)$ a formula that enumerates P^* in $\mathcal{S}$, a the Gödel number of $\forall v_2 \neg A(v_1, v_2)$ and G the sentence $\forall v_2 \neg A(\bar{a}, v_2)$. Then:

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If $\mathcal{S}$ is ω-consistent, then according to the ω-consistency lemma, $\neg\forall v_2 \neg A(v_1, v_2)$ represents P^*. So by the first Theorem of this class, G is undecidable.

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But ω-consistency is needed to the irrefutability of G only. If G is provable, then by a lemma of the previous class, $a \in P^*$. Because $A(v_1, v_2)$ enumerates P^*, there is an m s.t. $A(\bar{a}, \bar{m})$ is provable. Then $\exists v_2 A(\bar{a}, v_2)$, i.e $\neg G$ is provable, too. By consistency, G is not provable.