Tarski's theorem

András Máté

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The regular formula $F(v_1, v_2, \ldots v_n)$ <u>expresses</u> the relation (set of n-tuples of numbers) R iff

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A function f with n variables is Arithmetic iff the corresponding $n+1$-ary relation is Arithmetic.

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3. If f and A are Arithmetic/arithmetic, then $f^{-1}(A)$ is Arithmetic/arithmetic, too.
4. The composition of two one-variable Arithmetic functions is Arithmetic.
5. If A is an infinite Arithmetic set, then the relation $R(x, y)$ which holds iff x is the smallest element of A larger than y is Arithmetic.

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Decimal case:

$$m *_{10} n = m \cdot 10^{l(n)} + n$$

where $l(n)$ is the length of n (in the decimal system).

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5. $x *_b y = z \leftrightarrow x \cdot b^{l_b(y)} + y = z \leftrightarrow$
   $\exists z_1 \exists z_2 (b^{l_b(y)} = z_1 \wedge x \cdot z_1 = z_2 \wedge z_2 + y = z)$
   is Arithmetic, q.e.d.

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Proof: by induction on $n \geq 2$.

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Gödel numbers of the letters:

$$\begin{array}{ccccccccccccc} 0 & ' & (&) & f & , & v & \neg & \rightarrow & \forall & = & \leq & \sharp \\ 1 & 0 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & \eta & \varepsilon & \delta \end{array}$$

where $\eta, \varepsilon, \delta$ are 13-ary digits having the value 10, 11, 12.

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For the second: the Gödel number of $\bar{n}$ (0 followed by n strokes is $10\ldots0_{13}$ (where the number of 0-s is n), i.e. 13^n.

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For any expression E, $\forall v_1(v_1 = \bar{n} \rightarrow E)$ is an expression $E[\bar{n}]$ – this is how our application function (Φ) is given.

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Therefore, $r(x, y) = z$ iff $z = k * 13^y * 8 * x * 3$. (k is a particular number, you can calculate it.)

The diagonal function of $\mathcal{L}_E$ is $d(x) = r(x, x)$. $d(n)$ is the Gödel number of $E_n[\bar{n}]$.

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Lemma: If A is Arithmetic, then $A^* = d^{-1}(A)$ is Arithmetic, too.

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Let $D(v_1, v_2)$ express the function d, $F(v_1)$ express the set A. Then $\exists v_2(D(v_1, v_2) \wedge F(v_2))$ expresses A^*.

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$$H(\bar{h}) \text{ is true } \leftrightarrow h \in A^* \longleftrightarrow d(h) \in A$$

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Tarski's theorem: The set of the Gödel numbers of true sentences T is not Arithmetic.

Let us suppose (indirect assumption) that $F(v_1)$ expresses T. Then $\neg F(v_1)$ expresses $\tilde{T}$. Hence, $\tilde{T}$ has a Gödel sentence G. G is true iff its Gödel number is not in T i.e. iff it is false. Contradiction. Therefore there cannot be such an $F(v_1)$, q.e.d.

Excercises for homework

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1. Let us work with 10-based Gödel numbering, let the Gödel numbers of $\rightarrow$, $\forall$, $=$, $\leq$, $\sharp$ be the numbers 89, 899, 8999, 89999, 89999. Find a Gödel sentence for the set of even numbers. Is this sentence true or not?

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2. Find an Arithmetic function f s.t. if n is a Gödel number of some formula $F(v_1)$, then $f(n)$ is the Gödel number of some Gödel sentence of the set expressed by $F(v_1)$.

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I. Schemes for propositional logic

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I. Schemes for propositional logic

L_1 $(F \to (G \to F))$

L_2 $((F \to (G \to H)) \to ((F \to G) \to (F \to H)))$

L_3 $((\neg F \to \neg G) \to (G \to F))$

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I. Schemes for propositional logic

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- L_2 $((F \to (G \to H)) \to ((F \to G) \to (F \to H)))$
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II.Schemes for FOL with identity

- L_4 $(\forall x(F \to G) \to (\forall x F \to \forall x G))$
- L_5 $(F \to \forall x F)$ provided x has no occurrence in F.

Axioms for Peano Arithmetic with Exponentiation (P.E.)

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- L_5 $(F \to \forall x F)$ provided x has no occurrence in F.
- L_6 $\exists x(x = t)$ provided x has no occurrence in t.
- L_7 $(x = t \to (X_1 x X_2 \to X_1 t X_2))$, where X_1 and X_2 are expressions s.t. $X_1 x X_2$ is an atomic formula.

Arithmetic axioms

III. Axioms for arithmetic operations

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N_8 $(v_1 \leq v_2' \leftrightarrow (v_1 \leq v_2 \vee v_1 = v_2'))$

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- N_8 $(v_1 \leq v_2' \leftrightarrow (v_1 \leq v_2 \vee v_1 = v_2'))$
- N_9 $(v_1 \leq v_2 \vee v_2 \leq v_1)$
- N_{10} $(v_1 \mathbf{E} \bar{0}) = \bar{0}'$
- N_{11} $(v_1 \mathbf{E} v_2') = ((v_1 \mathbf{E} v_2) \cdot v_1)$

Mathematical induction

IV. Axiom scheme for mathematical induction

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Inference rules and proofs

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A formula F is provable in P.E. iff there is a proof whose last member is F.

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In addition, the $n+1$-ary relation

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- $x\tilde{P}y$ instead of $\neg xPy$.