Gödel's Incompleteness theorems

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András Máté

Eötvös Loránd University Budapest
Institute of Philosophy, Department of Logic
mate.andras53@gmail.com

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- Source (= textbook):
  Smullyan, R., *Gödel's Incompleteness Theorems*, Oxford: Oxford U.P., 1992.

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- Necessary preliminary knowledge: first-order logic.
- Method: lecture + solving problems
- Evaluation: solving problems (during the classes or in the exam period).
- Webpage: http://phil.elte.hu/mate/incompl/incompl.html. Presentations (pdf-s) will be published after the classes.

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This machine has a Gödelian(-Tarskian) property.

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7. $\mathcal{T} \subseteq \mathcal{S}$: true sentences.

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$$n \in A^* \leftrightarrow d(n) \in A$$

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$d(h)$ is the Gödel number of $H(h)$. Hence, $H(h)$ is true iff it is not provable.

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The sentence E_n is a <u>Gödel sentence</u> for the set A iff

$$E_n \in \mathcal{T} \leftrightarrow n \in A$$

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ii. follows from i. and G_1.

Homework:

1. Prove that if our toy machine prints only true sentences, then it cannot print all the true sentences of the given alphabet.

2. Prove **GT** from **D**.