A Paraconsistent View on B and S5

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Paraconsistency

- In classical logic (and most other logics), the explosive non-contradiction principle

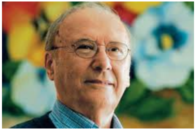
$$\varphi, \neg\varphi \vdash \psi$$

  allows us to derive any formula out of a contradiction. This makes any inconsistent theory trivial, and so no sensible reasoning can take place in the presence of contradictions.
- Paraconsistent logics do allow non-trivial inconsistent theories, i.e., in a logic **L** there are formulas φ, ψ, such that

$$\varphi, \neg\varphi \nvdash_{\mathbf{L}} \psi$$

The fathers of paraconsistent logic

S. Jaśkowski, 1948:
...PL should be rich enough to enable practical inferences.



N.C.A. da Costa, 1963:
...PL should contain as much as possible of classical logic.

The Brazilian School: C-systems

Definition

Let **L** be a logic for $\mathcal{L}$. A (primitive or defined) connective $\circ$ of **L** is a *consistency operator* with respect to $\neg$ if:

(b) $\vdash_{\mathbf{L}} (\circ\psi \wedge \neg\psi \wedge \psi) \supset \varphi$ for every $\psi, \varphi \in \mathcal{W}(\mathcal{L})$.

$\circ$ is a *strong consistency operator* if it is a consistency operator which satisfies also **(k)** $\circ\psi \vee (\neg\psi \wedge \psi)$ for every $\psi \in \mathcal{W}(\mathcal{L})$.

Definition

L *is a* C-system *if it is paraconsistent and has a strong consistency operator* $\circ$*.*

The basic C-system: BK

Definition

The logic **BK** is obtained by extending $\mathbf{CL}^+$ with the axioms **(b)** and **(k)**.

Family of C-systems: extensions of **BK** with various subsets of the following axioms:

(c)	$\neg\neg\varphi \supset \varphi$	**(e)**	$\varphi \supset \neg\neg\varphi$
$(\mathbf{n}^{\mathbf{l}}_{\wedge})$	$\neg(\varphi \wedge \psi) \supset (\neg\varphi \vee \neg\psi)$	$(\mathbf{n}^{\mathbf{r}}_{\wedge})$	$(\neg\varphi \vee \neg\psi) \supset \neg(\varphi \wedge \psi)$
$(\mathbf{n}^{\mathbf{l}}_{\vee})$	$\neg(\varphi \vee \psi) \supset (\neg\varphi \wedge \neg\psi)$	$(\mathbf{n}^{\mathbf{r}}_{\vee})$	$(\neg\varphi \wedge \neg\psi) \supset \neg(\varphi \vee \psi)$
$(\mathbf{n}^{\mathbf{l}}_{\supset})$	$\neg(\varphi \supset \psi) \supset (\varphi \wedge \neg\psi)$	$(\mathbf{n}^{\mathbf{r}}_{\supset})$	$(\varphi \wedge \neg\psi) \supset \neg(\varphi \supset \psi)$
$(\mathbf{o}^{\mathbf{1}}_{\sharp})$	$\circ\varphi \supset \circ(\varphi\sharp\psi)$	$(\mathbf{o}^{\mathbf{2}}_{\sharp})$	$\circ\psi \supset \circ(\varphi\sharp\psi)$
$(\mathbf{a}_{\sharp})$	$(\circ\varphi \wedge \circ\psi) \supset \circ(\varphi\sharp\psi)$	$(\mathbf{a}_{\neg})$	$\circ\varphi \supset \circ\neg\varphi$
(l)	$\neg(\varphi \wedge \neg\varphi) \supset \circ\varphi$	**(d)**	$\neg(\neg\varphi \wedge \varphi) \supset \circ\varphi$
$(\mathbf{i_1})$	$\neg\circ\varphi \supset \varphi$	$(\mathbf{i_2})$	$\neg\circ\varphi \supset \neg\varphi$

Motivation

Replacement Property

Let $\mathbf{L} = \langle \mathcal{L}, \vdash_{\mathbf{L}} \rangle$ be a logic.

- Formulas $\psi, \varphi \in \mathcal{W}(\mathcal{L})$ are *equivalent* in $\mathbf{L}$, denoted by $\psi \dashv\vdash_{\mathbf{L}} \varphi$, if $\psi \vdash_{\mathbf{L}} \varphi$ and $\varphi \vdash_{\mathbf{L}} \psi$.
- Formulas $\psi, \varphi \in \mathcal{W}(\mathcal{L})$ are *congruent* (or *indistinguishable*) in $\mathbf{L}$, if for every formula σ and atom p it holds that $\sigma[\psi/p] \dashv\vdash_{\mathbf{L}} \sigma[\varphi/p]$.
- $\mathbf{L}$ has the *replacement property* if any two formulas which are equivalent in $\mathbf{L}$ are congruent in it.

Question: Which C-systems with "nice" negation have this property?

Propositional Logic

A pair $\langle \mathcal{L}, \vdash \rangle$, where $\mathcal{L}$ is a propositional language, and $\vdash$ is a relation between sets of formulas of $\mathcal{L}$ and formulas of $\mathcal{L}$ that satisfies:

Reflexivity:	if $\varphi \in \mathcal{T}$ then $\mathcal{T} \vdash \varphi$.
Monotonicity:	if $\mathcal{T} \vdash \varphi$ and $\mathcal{T} \subseteq \mathcal{T}'$ then $\mathcal{T}' \vdash \varphi$.
Transitivity:	if $\mathcal{T} \vdash B$ and $\mathcal{T}, B \vdash \varphi$ then $\mathcal{T} \vdash \varphi$.
Structurality:	$\mathcal{T} \vdash \varphi$ then $\sigma(\mathcal{T}) \vdash \sigma(\varphi)$
Consistency	$p \nvdash q$

Positive Fragment of CL

$$\mathcal{L}_{CL^+} = \{\wedge, \vee, \supset\}$$

$\mathbf{IL}^+$ is the minimal logic $\mathbf{L}$ in $\mathcal{L}_{CL^+}$ such that:

- $\mathcal{T} \vdash_{\mathbf{L}} A \supset B$ iff $\mathcal{T}, A \vdash_{\mathbf{L}} B$
- $\mathcal{T} \vdash_{\mathbf{L}} A \wedge B$ iff $\mathcal{T} \vdash_{\mathbf{L}} A$ and $\mathcal{T} \vdash_{\mathbf{L}} B$
- $\mathcal{T}, A \vee B \vdash_{\mathbf{L}} C$ iff $\mathcal{T}, A \vdash_{\mathbf{L}} C$ and $\mathcal{T}, B \vdash_{\mathbf{L}} C$

$\mathbf{CL}^+$ is $\mathbf{IL}^+$ extended with the axiom $A \vee (A \supset B)$.

$\neg$-classical Logics

$$\mathcal{L}_{CL} = \{\wedge, \vee, \supset, \neg\}$$

A propositional logic $\mathbf{L} = \langle \mathcal{L}, \vdash_{\mathbf{L}} \rangle$ is **$\neg$-classical** if

- $\mathcal{L}_{CL} \subseteq \mathcal{L}$
- the $\mathcal{L}_{CL^+}$-fragment of $\mathbf{L}$ is $\mathbf{CL}^+$
- $\mathbf{L}$ satisfies:
    - $\mathcal{T} \vdash_{\mathbf{L}} A \supset B$ iff $\mathcal{T}, A \vdash_{\mathbf{L}} B$
    - $\mathcal{T} \vdash_{\mathbf{L}} A \wedge B$ iff $\mathcal{T} \vdash_{\mathbf{L}} A$ and $\mathcal{T} \vdash_{\mathbf{L}} B$
    - $\mathcal{T}, A \vee B \vdash_{\mathbf{L}} C$ iff $\mathcal{T}, A \vdash_{\mathbf{L}} C$ and $\mathcal{T}, B \vdash_{\mathbf{L}} C$

Paraconsistent Logics

A $\neg$-classical logic is **paraconsistent** if $\nvdash_{\mathbf{L}} (p \wedge \neg p) \supset q$.

Strongly Paraconsistent Logics

A ¬-classical logic is **strongly paraconsistent** if:

- $\nvdash_{\mathbf{L}} (p \wedge \neg p) \supset \neg q$
- $\nvdash_{\mathbf{L}} p \supset \neg p$
- $\nvdash_{\mathbf{L}} \neg p \supset p$.

Negation Properties

- $\neg$ is **complete** : $\mathcal{T} \vdash_{\mathbf{L}} \varphi$ whenever $\mathcal{T}, \psi \vdash_{\mathbf{L}} \varphi$ and $\mathcal{T}, \neg\psi \vdash_{\mathbf{L}} \varphi$.
- $\neg$ is **right-involutive**: $\varphi \vdash_{\mathbf{L}} \neg\neg\varphi$.
- $\neg$ is **left-involutive**: $\neg\neg\varphi \vdash_{\mathbf{L}} \varphi$.
- $\neg$ is **contrapositive**: $\neg\varphi \vdash_{\mathbf{L}} \neg\psi$ whenever $\psi \vdash_{\mathbf{L}} \varphi$.

Can't have it all

Proposition.

A $\neg$-classical logic in which $\neg$ is complete, right-involutive, and contrapositive cannot be strongly paraconsistent.

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$p \wedge \neg p \vdash_{\mathbf{L}} p$ and $p \wedge \neg p \vdash_{\mathbf{L}} \neg p$

By contrapositivity, $\neg p \vdash_{\mathbf{L}} \neg(p \wedge \neg p)$.

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Then $p \wedge \neg p \vdash_{\mathbf{L}} \neg(p \wedge \neg p)$

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Obviously, $\neg(p \wedge \neg p) \vdash_{\mathbf{L}} \neg(p \wedge \neg p)$

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Obviously, $\neg(p \wedge \neg p) \vdash_{\mathbf{L}} \neg(p \wedge \neg p)$

By completeness, $\vdash_{\mathbf{L}} \neg(p \wedge \neg p)$ and so also $q \vdash_{\mathbf{L}} \neg(p \wedge \neg p)$

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By completeness, $\vdash_{\mathbf{L}} \neg(p \wedge \neg p)$ and so also $q \vdash_{\mathbf{L}} \neg(p \wedge \neg p)$

By contrapositivity, $\neg\neg(p \wedge \neg p) \vdash_{\mathbf{L}} \neg q$

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Then $p \wedge \neg p \vdash_{\mathbf{L}} \neg(p \wedge \neg p)$
Obviously, $\neg(p \wedge \neg p) \vdash_{\mathbf{L}} \neg(p \wedge \neg p)$
By completeness, $\vdash_{\mathbf{L}} \neg(p \wedge \neg p)$ and so also $q \vdash_{\mathbf{L}} \neg(p \wedge \neg p)$
By contrapositivity, $\neg\neg(p \wedge \neg p) \vdash_{\mathbf{L}} \neg q$
By right-involutiveness, $p \wedge \neg p \vdash_{\mathbf{L}} \neg q$

What can we have?

CLuN : CL^+ and [t] $\neg\psi \vee \psi$ (completeness)
(Batens, 1998)

C_{min}: *CLuN* and [c] $\neg\neg\psi \supset \psi$ (completeness and left-involutivity)
(Carnielli, Coniglio and Marcos, 2007)

...and either right-involutivity or contrapositivity (BUT NOT BOTH!)

Can we construct a C-system with replacement property (and nice negation)?

Possible solution: adding axioms that ensure replacement condition:

$$\varphi \supset \psi, \psi \supset \varphi \vdash_{\mathbf{L}} \sigma[\psi/p] \supset \sigma[\varphi/p]$$

Proposition

Let **CAR** be the logic which is obtained from **CLuN** by adding $(\psi \supset \varphi) \wedge (\varphi \supset \psi) \supset (\neg\psi \supset \neg\varphi)$ as axiom. Then **CAR** is not strongly paraconsistent.

Second Attempt

- Refinement: allow inference of of $\neg\varphi \supset \neg\psi$ from $\varphi \supset \psi$ and $\psi \supset \varphi$ only when the premises are **theorems**.
- This can be done by including this rule in the corresponding proof systems not as a rule of derivation, but just as a **rule of proof**.
- *Rule of proof: a rule that is used only to define the set of axioms of the system, but not its consequence relation.*
- To make $\neg$ also contrapositive, we will adopt as a rule of proof the inference of $\neg\varphi \supset \neg\psi$ from just $\psi \supset \varphi$.

Contrapositivity and Replacement

Reminder

L has the *replacement property* if any two formulas which are equivalent in **L** are congruent in it.

Proposition

Let **L** be a $\neg$-classical logic in $\mathcal{L}_{CL}$ which extends $\mathbb{L}^+$, in which $\vdash_{\mathbf{L}} \neg\varphi \supset \neg\psi$ whenever $\vdash_{\mathbf{L}} \psi \supset \varphi$. Then **L** has the replacement property.

The logic NB

$Th(NB)$ is the minimal set S of formulas in $\mathcal{L}_{CL}$, such that:

1. S includes all axioms of HC_{min}.
2. S is closed under [MP] and the following rule:

$$[\mathrm{CP}] \quad \frac{\vdash \psi \supset \varphi}{\vdash \neg\varphi \supset \neg\psi}$$

Definition

HNB is the Hilbert-type system whose set of axioms is $Th(NB)$ and has [MP] for $\supset$ as its sole rule of inference.

Properties of NB

- Minimal extension of $\mathbf{CL}^+$ in $\mathcal{L}_{CL}$ in which $\neg$ is complete, contrapositive, and left-involutive
- strongly paraconsistent
- has the replacement property
- decidable
- is a C-system
- is the modal logic **B** in disguise!

A Gentzen-style System for NB

- The system *GNB* is obtained from *LK* by replacing ($[\neg\Rightarrow]$) by:

$$[\neg\Rightarrow]_B \quad \frac{\Gamma, \neg\Delta \Rightarrow \psi}{\neg\psi \Rightarrow \neg\Gamma, \Delta}$$

  (version of system proposed in Takano'92 and studied in Wansing'02.)

- *GNB* does not admit cut-elimination:
  $\vdash_{GNB} \neg(p \vee q), \neg(p \vee q) \to r \Rightarrow r$, but no cut-free proof.
- However, a weaker version of cut-elimination does hold, and implies decidability of **NB**.

Kripke-style Semantics for NB

$\langle W, R, \nu \rangle$ is called a **NB**-*frame* for $\mathcal{L}_{CL}$, if:

- W is a nonempty (finite) set (of "worlds")
- R is a reflexive and symmetric relation on W
- $\nu : W \times \mathcal{W}(\mathcal{L}_{CL}) \to \{t, f\}$ satisfies the following conditions:
  - $\nu(w, \psi \wedge \varphi) = t$ iff $\nu(w, \psi) = t$ and $\nu(w, \varphi) = t$.
  - $\nu(w, \psi \vee \varphi) = t$ iff $\nu(w, \psi) = t$ or $\nu(w, \varphi) = t$.
  - $\nu(w, \psi \supset \varphi) = t$ iff $\nu(w, \psi) = f$ or $\nu(w, \varphi) = t$.
  - $\nu(w, \neg\psi) = t$ iff there exists $w' \in W$ such that wRw', and $\nu(w', \psi) = f$.

Truth and Validity

Definition

Let $\langle W, R, \nu \rangle$ be a **NB**-frame.

- A formula φ is *true* in a world $w \in W$ ($w \Vdash \varphi$) if $\nu(w, \varphi) = t$.
- A sequent $s = \Gamma \Rightarrow \Delta$ is *true* in a world $w \in W$ ($w \Vdash s$) if $\nu(w, \varphi) = f$ for some $\varphi \in \Gamma$, or $\nu(w, \varphi) = t$ for some $\varphi \in \Delta$.
- A formula φ is *valid* in $\langle W, R, \nu \rangle$ ($\langle W, R, \nu \rangle \models \varphi$) if it is true in every world $w \in W$.
- A sequent s is *valid* in $\langle W, R, \nu \rangle$ ($\langle W, R, \nu \rangle \models s$) if it is true in every world $w \in W$.

Semantic Consequence

Definition

- Let $\mathcal{T} \cup \{\varphi\}$ be a set of formulas in $\mathcal{L}_{CL}$. φ *semantically follows in* **NB** from $\mathcal{T}$ if for every **NB**-frame $\langle W, R, \nu \rangle$ and every $w \in W$: if $w \Vdash \psi$ for every $\psi \in \mathcal{T}$ then $w \Vdash \varphi$.
- Let $S \cup \{s\}$ be a set of sequents in $\mathcal{L}_{CL}$. s *semantically follows in* **NB** from S if for every **NB**-frame $\mathcal{W}$, if $\mathcal{W} \models s'$ for every $s' \in S$, then $\mathcal{W} \models s$. s is **NB**-*valid* if s semantically follows in **NB** from $\emptyset$ (that is, s is valid in every **NB**-frame).

Completeness and Analyticity

Definition

A proof in G of s from S is called *analytic* if every formula occurring in it belongs to the set of subformulas of formulas in $S \cup \{s\}$.

Theorem

If s semantically follows in **NB** from S then s has an analytic proof in GNB from S.

Corollary

NB is decidable.

NB is a C-system

Reminder:

A *strong consistency operator* with respect to $\neg$ satisfies:

- **(b)** $\vdash_{\mathbf{L}} (\circ\psi \wedge \neg\psi \wedge \psi) \supset \varphi$ for every $\psi, \varphi \in \mathcal{W}(\mathcal{L})$.
- **(k)** $\vdash_{\mathbf{L}} \circ\psi \vee (\neg\psi \wedge \psi)$

NB has a strong consistency operator, which is **unique** (up to congruence):

$$\circ\varphi =_{def} (\varphi \wedge \neg\varphi) \supset \neg(\varphi \supset \varphi)$$

NB and axioms of C-systems

(c)	$\neg\neg\varphi \supset \varphi$
$(\mathbf{n}^{\mathbf{l}}_{\wedge})$	$\neg(\varphi \wedge \psi) \supset (\neg\varphi \vee \neg\psi)$
$(\mathbf{n}^{\mathbf{l}}_{\vee})$	$\neg(\varphi \vee \psi) \supset (\neg\varphi \wedge \neg\psi)$
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$(\mathbf{a}_{\vee})$	$(\circ\varphi \wedge \circ\psi) \supset \circ(\varphi \vee \psi)$
(l)	$\neg(\varphi \wedge \neg\varphi) \supset \circ\varphi$
$(\mathbf{i}_1)$	$\neg\circ\varphi \supset \varphi$

(e)	$\varphi \supset \neg\neg\varphi$
$(\mathbf{n}^{\mathbf{r}}_{\wedge})$	$(\neg\varphi \vee \neg\psi) \supset \neg(\varphi \wedge \psi)$
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$(\mathbf{n}^{\mathbf{r}}_{\supset})$	$(\varphi \wedge \neg\psi) \supset \neg(\varphi \supset \psi)$
$(\mathbf{a}_{\neg})$	$\circ\varphi \supset \circ\neg\varphi$
$(\mathbf{a}_{\supset})$	$(\circ\varphi \wedge \circ\psi) \supset \circ(\varphi \supset \psi)$
(d)	$\neg(\neg\varphi \wedge \varphi) \supset \circ\varphi$
$(\mathbf{i}_2)$	$\underline{\neg\circ\varphi \supset \neg\varphi}$

NB is the modal logic B!

- Modal logic **B**:
  - The language of **B** is usually taken to be $\{\wedge, \vee, \supset, \mathrm{F}, \Box\}$ (or $\{\wedge, \vee, \supset, \neg, \Box\}$, where $\neg$ denotes the *classical* negation).
  - Its semantics is given by Kripke frames:
    - accessibility relation R - reflexive and symmetric
    - notion of a 'Kripke frame' is defined like in **NB**, except that instead of the clause there for $\neg$ we have a clause for $\Box$:

      $\nu(w, \Box\psi) = t$ iff $\nu(w', \psi) = t$ for every $w' \in W$ s.t. wRw'.
- Languages of **B** and **NB** have the same expressive power, and $\neg$ and $\Box$ are interdefinable:
  - In the language of **NB**:

    $\Box\varphi =_{def} \sim\neg\varphi$, where $\sim\psi =_{def} \psi \supset \mathrm{F}$ and $\mathrm{F} =_{def} \neg(p_1 \supset p_1)$.
  - In the language of **B**:

$$\neg\varphi =_{def} \sim\Box\varphi$$

Advantages of the new presentation of B

- **Simpler language**: **NB** really has only two basic connectives: $\supset$ and $\neg$, while the standard presentation of **B** needs $\supset$, F, and $\Box$.
- **Simpler Hilbert-style calculus**: the standard system for **B** is obtained from *HCL* by the addition of:
  - the necessitation rule (if $\vdash \varphi$ then $\vdash \Box\varphi$).
  - three axioms:
    - **(K)** $\Box(\varphi \supset \psi) \supset (\Box\varphi \supset \Box\psi)$
    - **(T)** $\Box\varphi \supset \varphi$
    - **(B)** $\varphi \supset \Box\Diamond\varphi$, where $\Diamond\varphi =_{def} \sim\Box\sim\varphi$.

  The system for **NB** is obtained by the addition of one rule of proof, and just two simple and natural axioms.

The axiom (i_2) $\neg \circ \varphi \supset \varphi$

- By adding the axiom **($\mathbf{i}_2$)** to **NB**, we obtain another interesting logic, **NS5**.
- Studied by Béziau (2002), Batens (2002) and Osorio et al (2014).
- **NS5** is a strongly paraconsistent decidable logic with a complete, left-involutive and contrapositive negation and the replacement property.
- **NS5** is equivalent to the famous **S5**.

Summary

- We studied two logics with the following properties:
  - paraconsistent and yet have a nice negation: complete, left-involutive and contrapositive.
  - decidable
  - enjoy the replacement property
  - provide alternative presentations of two famous modal logics.
- A general method of turning modal logics into paraconsistent C-systems by taking $\neg\psi =_{def} \sim \Box\psi$ (where $\sim$ is the classical negation).
- What other interesting paraconsistent logics can be obtained?

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- A general method of turning modal logics into paraconsistent C-systems by taking $\neg\psi =_{def} \sim \Box\psi$ (where $\sim$ is the classical negation).
- What other interesting paraconsistent logics can be obtained?
- Stay tuned: another investigation of paraconsistent logics from a modal viewpoint - upcoming talk by J. Marcos tomorrow (Lahav, Marcos and Zohar, 2016)...