Axiomatizing modal dependence logics

Fan Yang
Delft University of Technology, The Netherlands

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- Henkin quantifiers (1961), Independence-friendly logic (Hintikka, Sandu 1989)
- First-order dependence logic (Väänänen 2007)

- Modal dependence logic (Väänänen 2008):

Modal Logic $\quad + \quad =(\vec{p}, q)$

The value of q is *completely determined* by the values of $\vec{p}$.

SYNTAX:

- (Classical) modal logic:

$$\alpha ::= p \mid \bot \mid \neg\alpha \mid \alpha \wedge \alpha \mid \alpha \otimes \alpha \mid \Box\alpha \mid \Diamond\alpha$$

- Modal dependence logic (**MD**):

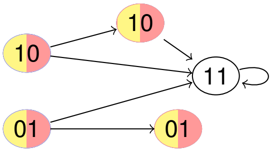
$$\phi ::= \alpha \mid =(p_1, \ldots, p_n, q) \mid \phi \wedge \phi \mid \phi \otimes \phi \mid \Box\phi \mid \Diamond\phi$$

- Modal downward closed team logic (**MT**):

$$\phi ::= \alpha \mid =(\alpha_1, \ldots, \alpha_n, \beta) \mid \phi \wedge \phi \mid \phi \otimes \phi \mid \phi \vee \phi \mid \phi \to \phi \mid \Box\phi \mid \Diamond\phi$$

Team Semantics (Hodges 1997)

- $\mathfrak{M} = (W, R, V)$
- A team $X \subseteq W$



$\mathfrak{M}, X \models {=}(\vec{\alpha}, \beta)$ iff for any $w, v \in X$,

$[\text{for all } i : \mathfrak{M}, w \models \alpha_i \Leftrightarrow \mathfrak{M}, v \models \alpha_i] \Longrightarrow [\mathfrak{M}, w \models \beta \Leftrightarrow \mathfrak{M}, v \models \beta]$

$\mathfrak{M}, X \models \alpha$ iff $\mathfrak{M}, w \models \alpha$ for all $w \in X$

$\mathfrak{M}, X \models \Diamond\phi$ iff there exists $Y \subseteq W$ such that XRY and $\mathfrak{M}, Y \models \phi$

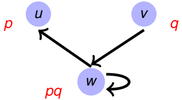
$\mathfrak{M}, X \models \Box\phi$ iff $\mathfrak{M}, R(X) \models \phi$

Fact: $\Box\alpha \equiv \neg\Diamond\neg\alpha$

(Empty Team Property) $\mathfrak{M}, \emptyset \models \phi$.

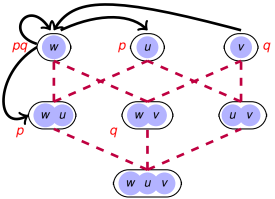
(Downward Closure Property) $\mathfrak{M}, X \models \phi$ and $Y \subseteq X \Longrightarrow \mathfrak{M}, Y \models \phi$.

A (modal) Kripke model $\mathfrak{M} = (W, R, V)$



The *powerset model* $\mathfrak{M}^\circ = (W^\circ, \supseteq, R^\circ, V^\circ)$ *induced by* $\mathfrak{M}$:

- $W^\circ = \wp(W) \setminus \{\emptyset\}$
- $XR^\circ Y$ iff XRY
- $X \in V^\circ(p)$ iff $X \subseteq V(p)$



Fact:

- $\mathfrak{M}^\circ$ is a bi-relation Kripke model of Fischer Servi's intuitionistic modal logic, and $\mathfrak{M}, X \models \phi \iff \mathfrak{M}^\circ, X \Vdash \phi$.

$$\mathfrak{M}, X \models \phi \to \psi \text{ iff for all } Y \subseteq X, \text{ if } \mathfrak{M}, Y \models \phi \text{ then } \mathfrak{M}, Y \models \psi$$
$$\mathfrak{M}, X \models \phi \lor \psi \text{ iff } \mathfrak{M}, X \models \phi \text{ or } \mathfrak{M}, X \models \psi$$

- $(W^\circ, \supseteq)$ is a Medvedev frame and a frame of Krisel-Putnam logic
- V° is a negative valuation: $\mathfrak{M}^\circ, X \Vdash \neg\neg p \iff \mathfrak{M}^\circ, X \Vdash p$

A Hilbert style system of **MT**

Axioms:

1. axioms of Fischer Servi's intuitionistic modal logic **IK**
   1. all axioms of **IPC**
   2. $\Box(\phi \to \psi) \to (\Box\phi \to \Box\psi)$
   3. $\Box(\phi \to \psi) \to (\Diamond\phi \to \Diamond\psi)$
   4. $\neg\Diamond\bot$
   5. $\Diamond(\phi \vee \psi) \to (\Diamond\phi \vee \Diamond\psi)$
   6. $(\Diamond\phi \to \Box\psi) \to \Box(\phi \to \psi)$
2. Krisel-Putnam axiom: $(\alpha \to (\phi \vee \psi)) \to (\alpha \to \phi) \vee (\alpha \to \psi)$
3. $\neg\neg\alpha \to \alpha$
4. $\neg\Diamond\neg\alpha \to \Box\alpha$
5. $\Box(\phi \vee \psi) \to (\Box\phi \vee \Box\psi)$
6. $=\!(\alpha_1, \ldots, \alpha_k, \beta) \leftrightarrow (\,=\!(\alpha_1) \wedge \cdots \wedge =\!(\alpha_k) \to =\!(\beta))$
7. $=\!(\alpha) \leftrightarrow (\alpha \vee \neg\alpha)$
8. $(\alpha \otimes \beta) \leftrightarrow (\neg\alpha \to \beta)$
9. $\phi \otimes (\psi \vee \chi) \leftrightarrow (\phi \otimes \psi) \vee (\phi \otimes \chi)$

Rules:

1. Modus Ponens: $\phi, \phi \to \psi / \psi$
2. Necessitation: $\phi / \Box\phi$

Theorem (Completeness)

$\models \phi \iff \vdash_{\mathbf{MT}} \phi$.

Proof. "$\Longrightarrow$":

$$\begin{array}{lll} & \models \phi \dashv\vdash \alpha_1 \vee \cdots \vee \alpha_n & \\ \Longrightarrow & \models \alpha_i \text{ for some } i & \text{(Disjunction Property)} \\ \Longrightarrow & \vdash_{\mathbf{K}} \alpha_i & \\ \Longrightarrow & \vdash_{\mathbf{MT}} \alpha_i & \\ \Longrightarrow & \vdash_{\mathbf{MT}} \phi & \end{array}$$

□

- $\phi ::= \alpha \mid =(p_1, \ldots, p_n, q) \mid \phi \wedge \phi \mid \phi \otimes \phi \mid \Box\phi \mid \Diamond\phi$

(Sano and Virtema 2015: Hilbert system, labeled tableau calculus)

A natural deduction system of modal dependence logic (**MD**)

1. **K** rules (with weaker rules for $\otimes$)
2. rules simulating $=(p) \equiv p \vee \neg p$ and $=(p, q) \equiv =(p) \to =(q)$

$$\frac{p}{=(p)} \qquad \frac{\neg p}{=(p)}$$

$$\frac{=(p) \quad \begin{matrix}[p]\\ \vdots \\ \theta\end{matrix} \quad \begin{matrix}[\neg p]\\ \vdots \\ \theta\end{matrix}}{\theta} \qquad \frac{\begin{matrix}[=(p)]\\ \vdots \\ =(q)\end{matrix}}{=(p,q)} \qquad \frac{=(p,q) \quad =(p)}{=(q)}$$

$$\frac{\phi \quad \begin{matrix}[\phi(p/[=(p), m])]\\ \vdots \\ \theta\end{matrix} \quad \begin{matrix}[\phi(\neg p/[=(p), m])]\\ \vdots \\ \theta\end{matrix}}{\theta}$$

(Y. and Väänänen, Propositional logics of dependence, APAL, 2016)

Theorem (Completeness)

$\models \phi \iff \vdash_{\mathbf{MD}} \phi$.

Proof. "$\Longrightarrow$":

$$\begin{array}{rl} & \models \phi(=(p)) \equiv \phi(p \vee \neg p) \equiv \phi(p) \vee \phi(\neg p) \\ \Longrightarrow & \text{w.l.o.g. } \models \phi(\neg p) \quad \text{(Disjunction Property)} \\ \Longrightarrow & \vdash_{\mathbf{K}} \phi(\neg p) \\ \Longrightarrow & \vdash_{\mathbf{MD}} \phi(\neg p) \\ \Longrightarrow & \vdash_{\mathbf{MD}} \phi(=(p)) \quad (\,\neg p / =(p)\,) \end{array}$$

□