"KNOWING VALUE" LOGIC AS A NORMAL MODAL LOGIC

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BACKGROUND

STANDARD EPISTEMIC LOGIC

Modal logics that reason about propositional knowledge (and belief) [von Wright 1951, Hintikka 1962]

- Language: “agent *i knows that* φ” ($K_i\varphi$).
- Semantics: you know that φ iff φ is true in all the epistemic alternatives that you cannot distinguish from the actual world.
- Proof systems: usually between S4 and S5.

$$s \vDash p \wedge \neg K_i p$$

BEYOND "KNOWING THAT"

Knowledge is not only expressed in terms of "knowing that":

- I *know whether* the claim is true.
- I *know what* your password is.
- I *know how* to go to Budapest.
- I *know why* he was late.
- I *know who* proved this theorem.

Hits (in millions) returned by google:

X	that	whether	what	how	who	why
"know X"	574	28	592	490	112	113
"knows X"	50.7	0.51	61.4	86.3	8.48	3.55

BEYOND "KNOWING THAT"

Knowledge is not only expressed in terms of "knowing that":

- I *know whether* the claim is true.
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- I *know how* to go to Budapest.
- I *know why* he was late.
- I *know who* proved this theorem.

Linguistically: factive verbs, embedded questions, exhaustivity
Philosophically: reducible to "knowledge-that"?
Logically: how to reason about "know-wh"?
Computationally: efficient representation and reasoning

BEYOND "KNOWING THAT": THE RESEARCH AGENDA

"knowing who" was discussed by Hintikka (1962) in terms of first-order modal logic: $\exists x K_i(John = x)$, i.e., knowing the answer of the embedded question. Asking a wh-question is to know.

Our "minimalistic" approach:

- Take a know-wh construction as a single modality, e.g., pack $\exists x K_i(John = x)$ into $Kwho_i John$.
- Balance the complexity and expressive power.
- Find intuitive reasoning patterns of different knowing X.
- New dynamics of knowledge (wait for Alexandru's talk).

BEYOND KNOWING THAT: (TECHNICAL) DIFFICULTIES

- (apparently) not normal:
  - $\nvdash Kw(p \to q) \land Kw\, p \to Kw\, q$
  - $\nvdash Khow\varphi \land Khow\psi \to Khow(\varphi \land \psi)$
  - $\vdash \varphi \not\Rightarrow \vdash Kwhy\varphi$
- combinations of quantifiers and modalities: $\exists x \Box \varphi(x)$;
- the axioms depend on the special schema of φ essentially;
- weak language vs. rich model: hard to axiomatize.

See *Beyond knowing that: a new generation of epistemic logic* for a survey on our logics of knowing whether, knowing what, and knowing how (`http://arxiv.org/abs/1605.01995`).

A DISGUISED NORMAL MODAL LOGIC

"KNOWING VALUE" OPERATOR KV PROPOSED BY [PLAZA 89]

The language **ELKv** is defined as (where $c \in C$):

$$\varphi ::= \top \mid p \mid \neg\varphi \mid (\varphi \wedge \varphi) \mid \mathcal{K}_i\varphi \mid \mathcal{K}v_i c$$

ELKv is interpreted on FO-epistemic models $\mathcal{M} = \langle S, D, \{\sim_i \mid i \in \mathbf{I}\}, V, V_C \rangle$ where D is a *constant* domain, V_C assigns to each (non-rigid) $c \in C$ a $d \in D$ on each $s \in S$:

$$\mathcal{M}, s \vDash \mathcal{K}v_i c \iff \text{for any } t_1, t_2: \text{ if } s \sim_i t_1, s \sim_i t_2, \text{ then } V_C(c, t_1) = V_C(c, t_2).$$

Essentially it is $\exists x K_i(c = x)$, which cannot be expressed by a finite disjunction in principle.

ELKv can express "*i* knows that *j* knows the password but *i* doesn't know what exactly it is" by $\mathcal{K}_i\mathcal{K}v_j c \wedge \neg\mathcal{K}v_i c$.

CONDITIONALLY KNOWING VALUE [WANG AND FAN IJCAI 2013]

We propose a conditional generalization of $\mathcal{K}v_i$ operator (call the language **ELKv**^{r}):

$$\varphi ::= \top \mid p \mid \neg\varphi \mid (\varphi \wedge \varphi) \mid \mathcal{K}_i\varphi \mid \mathcal{K}v_i(\varphi, c)$$

where $\mathcal{K}v_i(\varphi, c)$ says "agent i knows what c is, *given* φ".

$\mathcal{M}, s \vDash \mathcal{K}v_i(\varphi, c)$	$\Leftrightarrow$	for any $t_1, t_2 \in S$ such that $s \sim_i t_1$ and $s \sim_i t_2$: $\mathcal{M}, t_1 \vDash \varphi \& \mathcal{M}, t_2 \vDash \varphi$ implies $V_C(c, t_1) = V_C(c, t_2)$

Essentially $\exists x K_i(\varphi \to c = x)$ and $\mathcal{K}v_i c := \mathcal{K}v_i(\top, c)$. This language is equally expressive as **ELKv** with public announcements.

AXIOMATIZING ELKV^r OVER S5 FRAMES [WANG AND FAN AIML2014]

System $\mathbb{S}5\text{-}\mathbb{ELKVR}$

Axiom Schemas

TAUT	all the instances of tautologies
DISTK	$\mathcal{K}_i(p \to q) \to (\mathcal{K}_i p \to \mathcal{K}_i q)$
T	$\mathcal{K}_i p \to p$
4	$\mathcal{K}_i p \to \mathcal{K}_i \mathcal{K}_i p$
5	$\neg \mathcal{K}_i p \to \mathcal{K}_i \neg \mathcal{K}_i p$
DISTKv^r	$\mathcal{K}_i(p \to q) \to (\mathcal{K}v_i(q, c) \to \mathcal{K}v_i(p, c))$
Kv^{r}4	$\mathcal{K}v_i(p, c) \to \mathcal{K}_i \mathcal{K}v_i(p, c)$
Kv$^r\bot$	$\mathcal{K}v_i(\bot, c)$
Kv$^r\vee$	$\hat{\mathcal{K}}_i(p \wedge q) \wedge \mathcal{K}v_i(p, c) \wedge \mathcal{K}v_i(q, c) \to \mathcal{K}v_i(p \vee q, c)$

Rules

MP	$\dfrac{\varphi, \varphi \to \psi}{\psi}$
NECK	$\dfrac{\varphi}{\mathcal{K}_i \varphi}$
SUB	$\dfrac{\varphi}{\varphi[p/\psi]}$
RE	$\dfrac{\psi \leftrightarrow \chi}{\varphi \leftrightarrow \varphi[\psi/\chi}$

AXIOMATIZING ELKV^r OVER ARBITRARY FRAMES [DING 2015]

System $\mathbb{ELKVR}$

Axiom Schemas

TAUT	all the instances of tautologies
DISTK	$\mathcal{K}_i(p \to q) \to (\mathcal{K}_i p \to \mathcal{K}_i q)$
DISTKv^r	$\mathcal{K}_i(p \to q) \to (\mathcal{K}v_i(q, c) \to \mathcal{K}v_i(p, c))$
Kv$^r\bot$	$\mathcal{K}v_i(\bot, c)$
Kv$^r\vee$	$\hat{\mathcal{K}}_i(p \wedge q) \wedge \mathcal{K}v_i(p, c) \wedge \mathcal{K}v_i(q, c) \to \mathcal{K}v_i(p \vee q, c)$

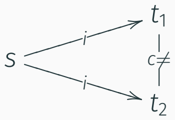
- The completeness proofs are highly non-trivial due to the imbalance between the rich model and limited language.
- The SAT problem of this logic is PSPACE-complete.
- Suitable bisimulation notion for this logic was unknown.

TWO QUESTIONS AND OUR KEY OBSERVATION

- How can it be connected to normal modal logic?
- How to rebalance the syntax and semantics?

Observation: $\neg\mathcal{K}v_i(\varphi, c)$ can be viewed as a special diamond:

$$\mathcal{M}, s \vDash \neg\mathcal{K}v_i(\varphi, c) \Leftrightarrow \text{there exist } t_1, t_2 \in S \text{ such that } s \sim_i t_1 \text{ and } s \sim_i t_2 : \mathcal{M}, t_1 \vDash \varphi \text{ and } \mathcal{M}, t_2 \vDash \varphi \text{ but } V_C(c, t_1) \neq V_C(c, t_2)$$



A MODAL LANGUAGE

To facilitate the comparison, we write $\neg\mathcal{K}v_i(\varphi, c)$ as $\Diamond_i^c\varphi$ and use the following language **MLKv**^{r}:

$$\varphi ::= \top \mid p \mid \neg\varphi \mid (\varphi \wedge \varphi) \mid \Box_i\varphi \mid \Diamond_i^c\varphi$$

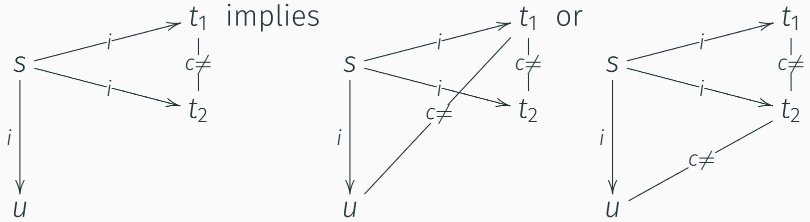
interpreted on Kripke models with binary and **ternary** relations $\langle S, \{\to_i : i \in I\}, \{R_i^c : i \in \mathbf{I}, c \in \mathbf{C}\}, V\rangle$, with extra conditions.

$$M, s \Vdash \Diamond_i^c\varphi \iff \exists\, u, v\text{: s.t. } sR_i^c uv \text{ and } \mathcal{M}, u \Vdash \varphi, \mathcal{M}, v \Vdash \varphi.$$

(1) $sR_i^c tu \iff sR_i^c ut$; (2) $sR_i^c uv$ only if $s \to_i u$ and $s \to_i v$;
(3) $sR_i^c tu$ and $s \to_i v$ implies that $sR_i^c tv$ or $sR_i^c uv$ holds;
(4) $sR_j^c tu$ for some $j \in \mathbf{I}$, $s \to_i t$ and $s \to_i u$ implies $sR_i^c tu$;
(5) $sR_j^c tu$ implies $t \neq u$.

AN INTERESTING PROPERTY

$sR_i^c t_1 t_2$ and $s \to_i u$ implies that at least one of $sR_i^c t_1 u$ and $sR_i^c t_2 u$ holds



We show that (4)(5) do not matter: For any set $\Gamma \cup \{\varphi\}$ of $\mathbf{MLKv}^r$ formulas: $\Gamma \Vdash_{\mathbb{C}_{1-5}} \varphi \iff \Gamma \Vdash_{\mathbb{C}_{1-3}} \varphi \iff t(\Gamma) \vDash t(\varphi)$ where t translates $\mathbf{MLKv}^r$ formulas back to $\mathbf{ELKv}^r$.

RECALL THE SYSTEM FOR ELKV^r.

System $\mathbb{ELKVR}$

Axiom Schemas

TAUT	all the instances of tautologies
DISTK	$\mathcal{K}_i(p \to q) \to (\mathcal{K}_i p \to \mathcal{K}_i q)$
DISTKv^r	$\mathcal{K}_i(p \to q) \to (\mathcal{K}v_i(q, c) \to \mathcal{K}v_i(p, c))$
Kv$^r\bot$	$\mathcal{K}v_i(\bot, c)$
Kv$^r\vee$	$\hat{\mathcal{K}}_i(p \wedge q) \wedge \mathcal{K}v_i(p, c) \wedge \mathcal{K}v_i(q, c) \to \mathcal{K}v_i(p \vee q, c)$

Rules

MP	$\dfrac{\varphi, \varphi \to \psi}{\psi}$
NECK	$\dfrac{\varphi}{\mathcal{K}_i\varphi}$
SUB	$\dfrac{\varphi}{\varphi[p/\psi]}$
RE	$\dfrac{\psi \leftrightarrow \chi}{\varphi \leftrightarrow \varphi[\psi/\chi]}$

In the new language:

- DISTKv^r: $\Box(p \to q) \to (\Box_i^c \neg q \to \Box_i^c \neg p)$ equivalent to $\Box(p \to q) \to (\Box_i^c p \to \Box_i^c q)$ under SUB and RE.
- Kv$^r\vee$: $\Diamond(p \wedge q) \wedge \Diamond_i^c(p \vee q) \to (\Diamond_i^c p \vee \Diamond_i^c q)$
- Kv$^r\bot$: $\Box_i^c \top$

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A NEW LOOK AT THE AXIOMATIZATION

System $\mathbb{SMLKVR}$

Axiom Schemas

TAUT	all the instances of tautologies
DISTK	$\Box_i(p \to q) \to (\Box_i p \to \Box_i q)$
DISTKv^r	$\Box_i(p \to q) \to (\Box_i^c p \to \Box_i^c q)$
Kv$^r\vee$	$\Diamond_i(p \wedge q) \wedge \Diamond_i^c(p \vee q) \to (\Diamond_i^c p \vee \Diamond_i^c q)$

Rules

MP	$\dfrac{\varphi, \varphi \to \psi}{\psi}$
NECK	$\dfrac{\varphi}{\Box_i \varphi}$
NECK^r	$\dfrac{\varphi}{\Box_i^c \varphi}$
RE	$\dfrac{\psi \leftrightarrow \chi}{\varphi \leftrightarrow \varphi[\psi/\chi]}$
SUB	$\dfrac{\varphi}{\varphi[p/\psi]}$

We replace $\Box_i^c \top$ by a necessitation rule NECK^r.

Theorem

$\mathbb{SMLKVR}$ is sound and complete w.r.t. $\mathbb{C}_{1-3}$ (and $\mathbb{C}_{1-5}$).

A relatively easy canonical model construction suffices (3 pages).

A NEW LOOK AT THE AXIOMATIZATION

System $\mathbb{SMLKVR}$

Axiom Schemas

TAUT	all the instances of tautologies
DISTK	$\Box_i(p \to q) \to (\Box_i p \to \Box_i q)$
DISTKv^r	$\Box_i(p \to q) \to (\Box_i^c p \to \Box_i^c q)$
Kv$^r\vee$	$\Diamond_i(p \wedge q) \wedge \Diamond_i^c(p \vee q) \to (\Diamond_i^c p \vee \Diamond_i^c q)$

Rules

MP	$\dfrac{\varphi, \varphi \to \psi}{\psi}$
NECK	$\dfrac{\varphi}{\Box_i \varphi}$
NECK^r	$\dfrac{\varphi}{\Box_i^c \varphi}$
RE	$\dfrac{\psi \leftrightarrow \chi}{\varphi \leftrightarrow \varphi[\psi/\chi]}$
SUB	$\dfrac{\varphi}{\varphi[p/\psi]}$

Note that $\Diamond_i^c(\varphi \vee \psi) \to (\Diamond_i^c \varphi \vee \Diamond_i^c \psi)$ does not hold. Moreover, $\Box_i^c(\varphi \to \psi) \to (\Box_i^c \varphi \to \Box_i^c \psi)$ does not hold neither, thus the logic is **not** normal.

However, this is only the appearance.

DISGUISED NORMAL MODAL LOGIC

$\Diamond_i^c$ is essentially a **binary** diamond!

In MLKvr we only allow $\Diamond_i^c(\varphi, \varphi)$. Let MLKvb be the language with $\Diamond_i^c(\varphi, \psi)$.

$\Diamond_i^c(\varphi, \psi)$ has the standard semantics for (polyadic) normal modal logic:

$$M, s \Vdash \Diamond_i^c(\varphi, \psi) \iff \exists\, u, v\text{: s.t. } sR_i^c uv \text{ and } \mathcal{M}, u \vDash \varphi, \mathcal{M}, v \Vdash \psi.$$

THE GENERALIZATION DOES NOT INCREASE EXPRESSIVITY

Proposition

MLKvb *is equally expressive as* MLKvr *over* $\mathbb{C}_{1-3}$.

$\Diamond_i^c(\varphi, \psi)$ is equivalent to the disjunction of the following:

- $\Diamond_i^c \varphi \wedge \Diamond_i \psi$
- $\Diamond_i^c \psi \wedge \Diamond_i \varphi$
- $\Diamond_i \varphi \wedge \Diamond_i \psi \wedge \neg \Diamond_i^c \varphi \wedge \neg \Diamond_i^c \psi \wedge \Diamond_i^c (\varphi \vee \psi)$

A NORMAL POLYADIC MODAL LOGIC

System $\mathbb{SMLKVB}$

Axiom Schemas

TAUT	all the instances of tautologies
DISTK	$\Box_i(p \to q) \to (\Box_i p \to \Box_i q)$
DISTBK	$\Box_i^c(p \to q, r) \to (\Box_i^c(p, r) \to \Box_i^c(q, r))$
SYM	$\Box_i^c(p, q) \to \Box_i^c(q, p)$
INCL	$\Diamond_i^c(p, q) \to \Diamond_i p$
DISBK	$\Diamond_i^c(p, q) \wedge \Diamond_i r \to \Diamond_i^c(p, r) \vee \Diamond_i^c(q, r)$

Rules

MP	$\dfrac{\varphi, \varphi \to \psi}{\psi}$
NECK	$\dfrac{\varphi}{\Box_i \varphi}$
NECKvb	$\dfrac{\varphi}{\Box_i^c(\varphi, \psi)}$
SUB	$\dfrac{\varphi}{\varphi[p/\psi]}$

Theorem

$\mathbb{SMLKVB}$ is sound and complete w.r.t. $\mathbb{C}_{1-3}$ and $\mathbb{C}_{1-5}$.

$\mathbb{SMLKVB}$ can drive all the axioms in $\mathbb{SMLKVR}$.

THE COMPLETENESS PROOF IS NOW SIMPLY ROUTINE (ONE PAGE)

$\mathcal{M}^c = \langle S, \{\to_i : i \in \mathbf{I}\}, \{R_i^c : i \in \mathbf{I}, c \in \mathbb{C}\}, V \rangle$

- S is the set of all maximal $\mathbb{SMLKVB}$-consistent sets of `MLKvb` formulas,
- $s \to_i t \iff \{\varphi : \Box_i \varphi \in s\} \subseteq t$,
- $sR_i^c tu \iff (1)\ \{\varphi : \Box_i \varphi \in s\} \subseteq t \cap u$ and (2) for any $\Box_i^c(\varphi, \psi) \in s$, $\varphi \in t$ or $\psi \in u$.
- $V(s) = \{p : p \in s\}$.

`SYM`, `INCL`, and `DISBK` are canonical for the corresponding properties 1-3.

ELKVR AS A NORMAL MODAL LOGIC

ELKv^r can be viewed as a disguised normal modal logic!

Standard techniques apply:

- Canonical model for free.
- Bisimulation for free.
- ? Decision procedure

These will help us in solving problems about the original ELKv^r.

Definition (Bisimulation)

Let $\mathcal{M}_1 = \langle S_1, \{\to_i^1 : i \in I\}, \{R_i^c : i \in I, c \in \mathbb{C}\}, V_1\rangle$, $\mathcal{M}_2 = \langle S_2, \{\to_i^2 : i \in I, c \in \mathbb{C}\}, \{Q_i^c : i \in I\}, V_2\rangle$ be two models for **MLKvb** (also for **MLKv**^{r}). A $\mathbb{C}$-bisimulation between $\mathcal{M}_1$ and $\mathcal{M}_2$ is a non-empty binary relation $Z \subseteq S_1 \times S_2$ such that for all $s_1 Z s_2$, the following conditions are satisfied:

- **Inv** : $V_1(s_1) = V_2(s_2)$;
- **Zig** : $s_1 \to_i^1 t_1 \Rightarrow \exists t_2$ such that $s_2 \to_i^2 t_2$ and $t_1 Z t_2$;
- **Zag** : $s_2 \to_i^2 t_2 \Rightarrow \exists t_1$ such that $s_1 \to_i^1 t_1$ and $t_1 Z t_2$;
- **Kvb-Zig** : $s_1 R_i^c t_1 u_1 \Rightarrow \exists t_2, u_2 \in S_2$ such that $t_1 Z t_2$, $u_1 Z u_2$ and $s_2 Q_i^c t_2 u_2$;
- **Kvb-Zag** : $s_2 Q_i^c t_2 u_2 \Rightarrow \exists t_1, u_1 \in S_1$ such that $t_1 Z t_2$, $u_1 Z u_2$ and $s_1 R_i^c t_1 u_1$.

A SIMPER LOGIC

Plaza's unconditional language:

$$\varphi ::= \top \mid p \mid \neg\varphi \mid (\varphi \land \varphi) \mid \mathcal{K}_i\varphi \mid \mathcal{K}v_i c$$

is essentially:

$$\varphi ::= \top \mid p \mid \neg\varphi \mid (\varphi \land \varphi) \mid \Box_i\varphi \mid \Box_i^c\bot$$

System $\mathbb{SMLKV}$

Axiom Schemas

TAUT	all the instances of tautologies
DISTK	$\Box_i(p \to q) \to (\Box_i p \to \Box_i q)$
INCLT	$\Diamond_i^c\top \to \Diamond_i\top$

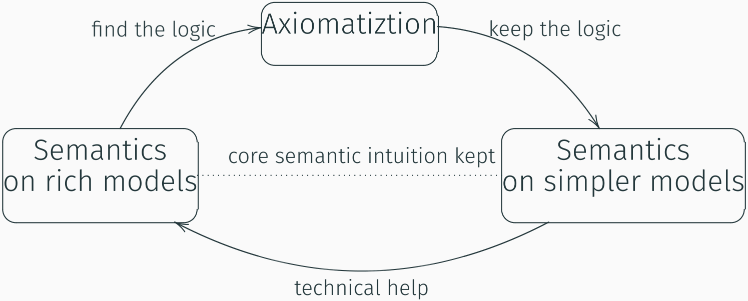
Rules

MP	$\dfrac{\varphi, \varphi \to \psi}{\psi}$
NECK	$\dfrac{\varphi}{\Box_i\varphi}$
SUB	$\dfrac{\varphi}{\varphi[p/\psi]}$
RE	$\dfrac{\psi \leftrightarrow \chi}{\varphi \leftrightarrow \varphi[\psi/\chi]}$

CONCLUSIONS

SIMPLIFY THE SEMANTICS WHILE KEEPING THE LOGIC

To restore the balance between the language and model:



SOME FUTURE DIRECTIONS

- Generalize it to other frames classes.
- Simplify the semantics for other knowing-X logics.

Thank you for your attention!

A survey paper on knowing-wh logics:
`http://arxiv.org/abs/1605.01995`.