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The Succinctness of First-order Logic over Modal Logic via a Formula Size Game

Lauri Hella & **Miikka Vilander**

School of Information Sciences,
University of Tampere, Finland

INTRODUCTION

Succinctness is an active research topic in modal logic.

Commonly used methods for proving succinctness results are **Adler-Immerman** (A-I) games and **extended syntax trees**.

In the A-I game, the second player has an easy optimal strategy, to always maximize their selections. The extended syntax tree on the other hand removes the game aspect entirely.

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Commonly used methods for proving succinctness results are **Adler-Immerman** (A-I) games and **extended syntax trees**.

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We present an alternative formula size game which has two players, neither of whom have an easy optimal strategy.

As an application of our game, we prove that first-order logic is non-elementarily more succinct than basic modal logic.

OUTLINE OF THE TALK

- Basics and notation
- Formula size game and how it works
- Succinctness of FO over ML

POINTED MODELS

Kripke-models are defined in the standard way.

We use pointed Kripke-models $(\mathcal{M}, w)$, where $w \in \mathrm{dom}(\mathcal{M})$.

Notation:

- $\Box(\mathcal{M}, w) := \{(\mathcal{M}, v) \mid v \in W, wR^{\mathcal{M}}v\}$.
- $\Box\mathbb{A} := \bigcup_{(\mathcal{M},w)\in\mathbb{A}} \Box(\mathcal{M}, w)$, where $\mathbb{A}$ is a set of pointed models.
- $\Diamond_f\mathbb{A} := f(\mathbb{A})$, where $f : \mathbb{A} \to \Box\mathbb{A}$ s.t. $f(\mathcal{M}, w) \in \Box(\mathcal{M}, w)$ for every $(\mathcal{M}, w) \in \mathbb{A}$.

We assume that all formulas of basic modal logic ML are in negation normal form.

Classes of pointed models

If $\mathbb{A}$ is a class of pointed Kripke-models, then

- $\mathbb{A} \models \varphi \Leftrightarrow (\mathcal{A}, w) \models \varphi$ for every $(\mathcal{A}, w) \in \mathbb{A}$,
- $\mathbb{A} \models \neg\varphi \Leftrightarrow (\mathcal{A}, w) \not\models \varphi$ for every $(\mathcal{A}, w) \in \mathbb{A}$.

A formula $\varphi \in \mathrm{ML}$ *separates the classes* $\mathbb{A}$ *and* $\mathbb{B}$ if

$$\mathbb{A} \models \varphi \text{ and } \mathbb{B} \models \neg\varphi.$$

n-BISIMILARITY

Two pointed Φ-models $(\mathcal{M}, w)$ and $(\mathcal{M}', w')$ are *n-bisimilar*, $(\mathcal{M}, w) \leftrightarrows_n (\mathcal{M}', w')$, if there are binary relations $Z_n \subseteq \cdots \subseteq Z_0$ such that for every $0 \leq i \leq n-1$ we have

(1) $(\mathcal{M}, w) Z_n (\mathcal{M}', w')$,

(2) if $(\mathcal{M}, v) Z_0 (\mathcal{M}', v')$, then $(\mathcal{M}, v) \models p \Leftrightarrow (\mathcal{M}', v') \models p$,

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(3) if $(\mathcal{M}, v) Z_{i+1} (\mathcal{M}', v')$ and $(\mathcal{M}, u) \in \Box(\mathcal{M}, v)$ then there is $(\mathcal{M}', u') \in \Box(\mathcal{M}', v')$ such that $(\mathcal{M}, u) Z_i (\mathcal{M}', u')$,

(4) if $(\mathcal{M}, v) Z_{i+1} (\mathcal{M}', v')$ and $(\mathcal{M}', u') \in \Box(\mathcal{M}', v')$ then there is $(\mathcal{M}, u) \in \Box(\mathcal{M}, v)$ such that $(\mathcal{M}, u) Z_i (\mathcal{M}', u')$.

FORMULA SIZE

Let $\varphi \in \mathrm{ML}$.

- *Modal size* $\mathrm{ms}(\varphi)$ is the number of modal operators in φ.
- *Connective size* $\mathrm{cs}(\varphi)$ is the number of **binary** connectives.
- *Size* $\mathrm{s}(\varphi) = \mathrm{ms}(\varphi) + \mathrm{cs}(\varphi)$.

Both $\mathrm{ms}(\varphi)$ and $\mathrm{cs}(\varphi)$ are defined recursively in the standard way.

Similarly the *size* of a formula $\psi \in \mathrm{FO}$, $s(\psi)$, is the number of binary connectives and quantifiers in ψ.

THE FORMULA SIZE GAME $\mathrm{FS}_{m_0,k_0}(\mathbb{A}_0, \mathbb{B}_0)$

- $\mathbb{A}_0$ and $\mathbb{B}_0$ are sets of pointed models.
- m_0 and k_0 are natural numbers.
- The two players are S and D.
- The starting position is $(m_0, k_0, \mathbb{A}_0, \mathbb{B}_0)$.
- Following positions of the game are of the form $(m, k, \mathbb{A}, \mathbb{B})$.
- The four different types of moves are left splitting move, right splitting move, left successor move and right successor move.

The splitting moves

Left splitting move: S chooses numbers m_1, m_2, k_1 and k_2 and sets $\mathbb{A}_1$ and $\mathbb{A}_2$ s.t. $m_1 + m_2 = m$, $k_1 + k_2 + 1 = k$ and $\mathbb{A}_1 \cup \mathbb{A}_2 = \mathbb{A}$. Then D decides whether the game continues from the position $(m_1, k_1, \mathbb{A}_1, \mathbb{B})$ or the position $(m_2, k_2, \mathbb{A}_2, \mathbb{B})$.

Right splitting move: Same as the left splitting move with the roles of $\mathbb{A}$ and $\mathbb{B}$ switched.

The successor moves

Left successor move: S chooses a function $f : \mathbb{A} \to \Box\mathbb{A}$ such that $f(\mathcal{A}, w) \in \Box(\mathcal{A}, w)$ for all $(\mathcal{A}, w) \in \mathbb{A}$.
The game continues from the position $(m-1, k, \Diamond_f \mathbb{A}, \Box\mathbb{B})$.

Right successor move: Same as the left successor move with the roles of $\mathbb{A}$ and $\mathbb{B}$ switched.

Winning the game

The game ends and S wins in a position $(m, k, \mathbb{A}, \mathbb{B})$ if there is a Φ-literal φ such that φ separates the sets $\mathbb{A}$ and $\mathbb{B}$.

The game ends and D wins in a position $(m, k, \mathbb{A}, \mathbb{B})$ if S cannot move and S does not win in this position.

Theorem

Let $\mathbb{A}$ and $\mathbb{B}$ be sets of pointed Φ-models and let m and k be natural numbers. Then the following conditions are equivalent:

$(\mathrm{win})_{m,k}$ *S has a winning strategy in the game $\mathrm{FS}_{m,k}(\mathbb{A}, \mathbb{B})$.*

$(\mathrm{sep})_{m,k}$ *There is a formula $\varphi \in \mathrm{ML}(\Phi)$ such that $\mathrm{ms}(\varphi) \leq m$, $\mathrm{cs}(\varphi) \leq k$ and the formula φ separates the sets $\mathbb{A}$ and $\mathbb{B}$.*

Theorem

Let $\mathbb{A}$ *and* $\mathbb{B}$ *be sets of pointed models and let* $m, k \in \mathbb{N}$. *If there are m-bisimilar pointed models* $(\mathcal{A}, w) \in \mathbb{A}$ *and* $(\mathcal{B}, v) \in \mathbb{B}$, *then D has a winning strategy for the game* $\mathrm{FS}_{m,k}(\mathbb{A}, \mathbb{B})$.

Theorem

Let $\mathbb{A}$ and $\mathbb{B}$ be sets of pointed models and let $m, k \in \mathbb{N}$. If there are m-bisimilar pointed models $(\mathcal{A}, w) \in \mathbb{A}$ and $(\mathcal{B}, v) \in \mathbb{B}$, then D has a winning strategy for the game $\mathrm{FS}_{m,k}(\mathbb{A}, \mathbb{B})$.

Idea of proof:

D can maintain the condition of having m'-bisimilar models in any following position $(m', k', \mathbb{A}', \mathbb{B}')$.

If S makes a successor move, then by m'-bisimilarity, the new position will contain $m' - 1$-bisimilar models.

If S makes a splitting move, D need only choose the position where the m'-bisimilar models are present.

A PROPERTY OF POINTED FRAMES

We define for each $n \in \mathbb{N}$ a property that is definable in FO with a non-elementarily shorter formula than in ML.

We consider the case $\Phi = \emptyset$.

The property $\mathbb{A}_n$:

For each $n \in \mathbb{N}$ we let $\mathbb{A}_n$ be the class of all pointed frames $(\mathcal{A}, w)$ such that $(\mathcal{A}, u) \leftrightarrows_n (\mathcal{A}, v)$ for all $(\mathcal{A}, u), (\mathcal{A}, v) \in \Box(\mathcal{A}, w)$,

$\mathbb{B}_n$ is the complement of $\mathbb{A}_n$.

For every $n \in \mathbb{N}$, the classes $\mathbb{A}_n$ and $\mathbb{B}_n$ can be separated in FO with a formula with size $\mathcal{O}(2^n)$.

Furthermore, the property $\mathbb{A}_n$ is bisimulation invariant, so by the van-Benthem Characterization Theorem the classes $\mathbb{A}_n$ and $\mathbb{B}_n$ can also be separated in ML.

It remains to find out how large this ML formula must be.

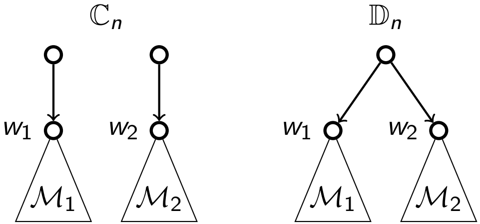
We define the function $\mathrm{twr} : \mathbb{N} \to \mathbb{N}$ recursively as follows:

$$\mathrm{twr}(0) = 1$$
$$\mathrm{twr}(n+1) = 2^{\mathrm{twr}(n)}.$$

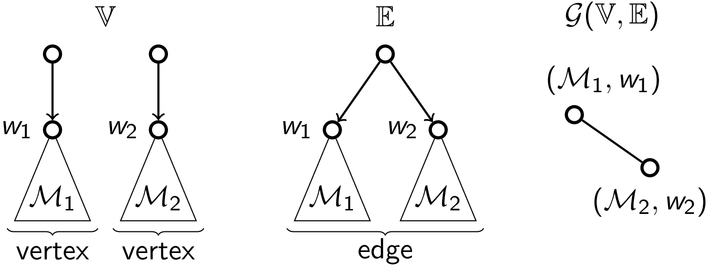
There are exactly $\mathrm{twr}(n)$ n-bisimilarity types of pointed models.

We define sets $\mathbb{C}_n \subseteq \mathbb{A}_n$ and $\mathbb{D}_n \subseteq \mathbb{B}_n$ for the formula size game.

If $(\mathcal{M}_1, w_1)$ and $(\mathcal{M}_2, w_2)$ are representatives of two different n-bisimilarity types, the depicted pointed frames are in the sets $\mathbb{C}_n$ and $\mathbb{D}_n$.



We associate to each pair of subsets $\mathbb{V} \subseteq \mathbb{C}_n$ and $\mathbb{E} \subseteq \mathbb{D}_n$ a graph $\mathcal{G}(\mathbb{V}, \mathbb{E})$ as depicted.



The splitting moves of the formula size game correspond to splitting the graph in two parts.

The left splitting move corresponds to splitting the vertex set of the graph and the right splitting move splits the edge set.

Successor moves lead to S losing the game as long as the graph has at least one edge.

We will use as an invariant in the formula size game the binary logarithm of the chromatic number of the graph $\mathcal{G}(\mathbb{V}, \mathbb{E})$.

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Lemma

Let $\mathcal{G} = (V, E)$ be a graph.

1. *Let $\emptyset \neq V_1, V_2 \subseteq V$ be s.t. $V_1 \cup V_2 = V$ and let $\mathcal{G}_1 = (V_1, E \restriction V_1)$ and $\mathcal{G}_2 = (V_2, E \restriction V_2)$. Then $\chi(\mathcal{G}) \leq \chi(\mathcal{G}_1) + \chi(\mathcal{G}_2)$.*
2. *Let $E_1, E_2 \subseteq E$ such that $E_1 \cup E_2 = E$ and let $\mathcal{G}_1 = (V, E_1)$ and $\mathcal{G}_2 = (V, E_2)$. Then $\chi(\mathcal{G}) \leq \chi(\mathcal{G}_1)\chi(\mathcal{G}_2)$.*

Lemma

Let $\emptyset \neq \mathbb{V} \subseteq \mathbb{C}_n$ *and* $\mathbb{E} \subseteq \mathbb{D}_n$ *for some* $n \in \mathbb{N}$ *and let* $m, k \in \mathbb{N}$*.*
If $\chi(\mathcal{G}(\mathbb{V}, \mathbb{E})) \geq 2$ *and* $k < \log(\chi(\mathcal{G}(\mathbb{V}, \mathbb{E})))$*, then D has a winning strategy in the game* $\mathrm{FS}_{m,k}(\mathbb{V}, \mathbb{E})$*.*

The graph $\mathcal{G}(\mathbb{C}_n, \mathbb{D}_n)$ is a complete graph and the vertex set is all $\mathrm{twr}(n)$ n-bisimilarity types so the chromatic number of this graph is $\mathrm{twr}(n)$.

Thus $\log(\chi(\mathcal{G}(\mathbb{C}_n, \mathbb{D}_n))) = \mathrm{twr}(n-1)$. The size of an ML formula separating $\mathbb{A}_n$ and $\mathbb{B}_n$ must therefore be at least $\mathrm{twr}(n-1)$.

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Theorem

Bisimulation invariant FO *is non-elementarily more succinct than* ML.

The classes $\mathbb{A}_n$ and $\mathbb{B}_n$ can also be separated in a suitable version ML_2 of two-dimensional modal logic with a formula of size $\mathcal{O}(2^n)$.

Thus we obtain a similar succinctness result for ML_2 and ML.

Theorem

The 2-dimensional modal logic ML_2 is nonelementarily more succinct than ML.

Thank you for your attention!