Temporal hybrid logics with the modalities «tomorrow» and «yesterday»

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Outline

1. Syntax and semantics
2. The case of discrete time
3. Results
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Syntax and semantics

Definition. *Hybrid temporal formulas HTF* are built from

1) $\mathrm{PROP} = \{p_1, p_2, \ldots\}$ — a countable set of propositional variables and
2) $\mathrm{N} = \{i_1, i_2, \ldots\}$ — a countable set of nominals,
such that $\mathrm{PROP} \cap \mathrm{N} = \oslash$

using the classical connectives $\rightarrow, \bot$
and the unary modal connectives $\bigcirc^+$ («tomorrow») and $\bigcirc^-$ («yesterday»).

Syntax and semantics

Definition. *Hybrid temporal formulas with the satisfaction operators* $HTF_{@}$ are built from PROP using
1) $\rightarrow, \bot$,
2)$\bigcirc^{+}$, $\bigcirc^{-}$
3) satisfaction operators $@_i$, $i \in \mathrm{N} = \{i_1, i_2, \ldots\}$

Definition. Let $F = (W, R^+, R^-)$ be a Kripke frame with two accessibility relations R^+, R^-. F is called an *SL.t-frame* if R^+ and R^- define two mutually inverse bijections — permutations on the set of worlds W:
1) $(R^+)^{-1} = R^-$
2) $\forall x \exists ! y \; xR^+y$
3) $\forall x \exists ! y \; xR^-y$

Lemma. $F = (W, R^+, R^-)$ is an *SL.t-frame* $\Leftrightarrow$ F validates the following set of (temporal) axioms (*):

(1) $\bigcirc^- \bigcirc^+ p \leftrightarrow p$,
(2) $\bigcirc^+ \bigcirc^- p \leftrightarrow p$,
(3)$\neg \bigcirc^+ p \leftrightarrow \bigcirc^+ \neg p$,
(4) $\neg \bigcirc^- p \leftrightarrow \bigcirc^- \neg p$.

Modal logic *SL.t* is also known as «yesterday»-«tomorrow» logic and as a logic with functional modalities.
SL.t was introduced in 1965 by Clifford and by Lemmon&Scott.
The first results about *SL.t* where published by Muchnik in 1979.

Temporal hybrid logics

Definition. $SL.t\mathcal{H}$

(1) all classical tautologies,

(2) the temporal axioms (*),

(3) the normality axioms:

$\bigcirc^{+}(p \to q) \to (\bigcirc^{+} p \to \bigcirc^{+} q)$, $\bigcirc^{-}(p \to q) \to (\bigcirc^{-} p \to \bigcirc^{-} q)$,

(4) the set of axioms (NOM):

$(i \wedge p) \to \bigcirc^{+n}(i \to p)$, $(i \wedge p) \to \bigcirc^{-n}(i \to p)$, for all $n \geq 1$

(where $\bigcirc^{+n} = \underbrace{\bigcirc^{+} \cdots \bigcirc^{+}}_{n \text{ times}}$ and $\bigcirc^{-n} = \underbrace{\bigcirc^{-} \cdots \bigcirc^{-}}_{n \text{ times}}$)

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(1) Modus Ponens,

(2) (Nec): if $A \in \Lambda$, then $\bigcirc^{+} A \in \Lambda$ and $\bigcirc^{-} A \in \Lambda$,

(3) (Subst): if $A(q) \in \Lambda$, where $q \in \mathrm{PROP}$, then $A(B) \in \Lambda$, for all $B \in \mathrm{HTF}$,

(4) (Subst′): if $i, j \in \mathrm{N}$ and $A(i) \in \Lambda$, then $A(j) \in \Lambda$.

Definition. $SL.t\mathcal{H}_{@}$

(1) all classical tautologies,
(2) the temporal axioms (*),
(3) the normality axioms,
(4) the following axioms:

$\neg @_i A \leftrightarrow @_i \neg A$,
$\bigcirc^{+} @_i A \rightarrow @_i A$,
$\bigcirc^{-} @_i A \rightarrow @_i A$,
$@_i @_j A \rightarrow @_j A$, where $i, j \in \mathrm{N}$

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(1) Modus Ponens,
(2) (Nec),
(3) (Subst) if $A(q) \in \Lambda$, where $q \in \mathrm{PROP}$, then $A(B) \in \Lambda$, for all $B \in \mathrm{HTF}_{@}$,
(4) ($Nec_{@}$) if $A \in \Lambda$, then $@_i A \in \Lambda$.

Semantics

Definition. Let F be an $SL.t$-frame.
A *hybrid Kripke model* on F is a pair (F, V),
where $V\colon \mathrm{PROP} \cup \mathrm{N} \to 2^W$ is such that $|V(i)| = 1$ for $i \in \mathrm{N}$.

Definition. The truth values of formulas in worlds of Kripke models are defined in a standard way:

$(M, w) \not\models \bot$,

$(M, w) \models i$ iff $w \in V(i)$,

$(M, w) \models p$ iff $w \in V(p)$,

$(M, w) \models A \to B$ iff $((M, w) \models A \Rightarrow (M, w) \models B)$,

$(M, w) \models \bigcirc^{+} A$ iff $\forall v \in W (wR^{+} v \Rightarrow (M, v) \models A)$,

$(M, w) \models \bigcirc^{-} A$ iff $\forall v \in W (wR^{-} v \Rightarrow (M, v) \models A)$,

$(M, w) \models @_i A$ iff $\forall v \in V(i)\ ((M, v) \models A)$.

Definition. Let $F = (W, R)$ be an $SL.t$-frame.
A set $\Gamma \subseteq HTF(HTF_{@}$ respectively) is *(hybrid) satisfiable in* F if it is true at a point of some hybrid Kripke model on F.

$A \in HTF(HTF_{@})$ is *(hybrid) valid on* F $(F \vDash A)$ if $\neg A$ is not satisfiable on F.

Completeness

Definition. A logic Λ is *complete for a class of frames* C if ($A \in \Lambda$ iff $C \models A$), for all formulas A.
A logic Λ is *strongly complete for a class of frames* C if $C \models \Lambda$ and for any Λ-consistent set of formulas Σ there is $F \in C$ such that Σ is satisfiable on F.
A logic Λ has *the finite model property (FMP)* if Λ is complete for some class of finite frames.
Theorem. $SL.t\mathcal{H}$ and $SL.t\mathcal{H}_{@}$ are strongly complete for the class of all $SL.t$-frames.

The case of discrete time

Definition. $\mathcal{HZ}.t = SL.t\mathcal{H} + (i \to \neg \bigcirc^{+n} i) + (i \to \neg \bigcirc^{-n} i)$, for all $n \geq 1$.

Definition. The frame $(\mathbb{Z}, R^+, R^-)$, where xR^+y if $y = x + 1$ and xR^-y if $y = x - 1$ is called the (integer) *line*.

Results

Theorem. $\mathcal{HZ}.t$ is strongly complete for a disjoint union of two lines: $(\mathbb{Z}, R^+, R^-) \sqcup (\mathbb{Z}, R^+, R^-)$.

Remark. The set of formulas $\{\neg \bigcirc^{+n} i \mid n \in \mathbb{Z}\}$ is $\mathcal{HZ}.t$-consistent, but is not satisfiable in a single line. So $\mathcal{HZ}.t$ cannot be strongly complete for a single line.

Results

Theorem 1.
(1)$\mathcal{HZ}.t$ is complete for a single line.
(2) $\mathcal{HZ}.t$ is antitabular (all frames are infinite) $\Rightarrow$ $\mathcal{HZ}.t$ lacks the FMP.
(3) The satisfability problem for $\mathcal{HZ}.t$ is NP-complete.
Theorem 2.
(1)$SL.t\mathcal{H}_{@}$ is complete for a single line.
(2)$SL.t\mathcal{H}_{@}$ has the FMP.
(3) The satisfability problem for $SL.t\mathcal{H}_{@}$ is NP-complete.

Problems

1) An interesting problem is to find properties of temporal hybrid logics without axioms $\bigcirc^{+}\neg p \rightarrow \neg \bigcirc^{+} p$ and $\bigcirc^{-}\neg p \rightarrow \neg \bigcirc^{-} p$ and of temporal hybrid logics of branching time.

2) We may also conjecture that $\mathcal{HZ}.t$ is Post complete in the class of $\mathcal{H}.t$-logics where the rule (Namelite) (if $\vdash \neg i$ then $\vdash \bot$) is admissible.

Thank you for your attention!

References

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Post completeness

Definition. A logic L is called Post complete in a lattice of logics if L is consistent and does not have proper consistent extensions in the lattice.

Definition. A logic L is called generally Post complete if it is consistent and does not have proper consistent extensions closed under the inference rules that are admissible in L.