Alexander Chagrov
1957–2016

Complexity of Approximability for Modal and Superintuitionistic Logics
Supervisor: Max Kanovich


Alexander Chagrov & Michael Zakharyaschev. Modal Logic


Modeling of computational processes by means of propositional logic

PhD Students: Mikhail Rybakov (2005), Igor Gorbunov (2006)
Decidability of properties of modal logics

**Problem:** given a modal logic \( L \), determine whether
- \( L \) is decidable
- \( L \) is Kripke complete
- \( L \) has the finite model property
- \( L \) has interpolation
- ...

**Kuznetsov (unpublished):** not-trivial properties of recursively axiomatisable modal logics are undecidable
(see Chagrov 1992)

**Thomason (1982):** Kripke completeness of finitely axiomatisable logics in NExt\( K \) is undecidable

But
- consistency is decidable in NExt\( K \) (Makinson 1971)
- interpolation is decidable in NExt\( S4 \) (Maksimova 1979)
- tabularity in NExt\( S4 \), NExt\( GL \) (Maksimova, Esakia & Meskhi, Blok)

**Chagrov & Chagrova (1990s):** a general method for showing undecidability of
- decidability, completeness, FMP, etc. in Ext\( Int \), NExt\( S4 \)
- interpolation in NExt\( GL \)
- first-order definability over \( S4 \) (2006: a simpler proof over \( K \))
- ...

AiML, Budapest, 2016
Decidability of properties of logics: known results

<table>
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<th>ExtInt</th>
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...
Independent axiomatisability

Old problem: does every normal modal or intermediate logic have an independent set of axioms?
(Citkin’s problem in Logic Notebook, 1986)

Blok’s problem (1980): are the lattices NExtS4 and ExtInt strongly coatomic?
every proper interval \([L_2, L_1]\) contains an immediate predecessor of \(L_1\)

Chagrov’s key observation: if \(L_1\) has an independent axiomatisation then,
for every finitely axiomatisable \(L_2 \subset L_1\),
there is an immediate predecessor of \(L_1\) in \([L_2, L_1]\)

Both ExtInt and NExtS4 contain logics without independent axiomatisations
these lattice are not strongly coatomic
(Chagrov & Zakharyaschev 1995) see open problems
Complexity problems (Chagrov’s PhD)

Complexity function for a logic $L$: $f_L(n) = \max \min |\mathcal{F}|$

$\mathcal{F} = \{ \phi \mid \phi \in L, \phi \not\in L \}$

Kuznetsov 1975: is $f_{\text{Int}} = \text{poly}(n)$? (or does Int have the polynomial FMP?)

in which case Int and Cl would be polynomially equivalent and NP = PSPACE

Chagrov 1985:

linear FMP: all logics containing S4.3

polynomial FMP: minimal logics of finite width in NExtK4 and ExtInt

minimal logics of finite depth NExtK4 and ExtInt

(all these logics are polynomially equivalent to Cl)

fast growing FMP: for any function $f(n)$, there are logics $L$ of width 1 in NExtK4 and of width 2 in ExtInt with FMP and such that $f_L(n) \geq f(n)$

Chagrov & M. Rybakov 2003: K4(0), Grz(1), GL(1) do not have polynomial FMP and are PSPACE- complete
Tabularity

Chagrov's tabularity criterion: a logic $L$ in Ext$K$ is tabular iff $\text{tab}_n \in L$, for some $n < \omega$.

$$\text{tab}_n = \alpha_n \land \text{alt}_n$$

$$\alpha_n = \neg(\varphi_1 \land \Diamond(\varphi_2 \land \Diamond(\varphi_3 \land \ldots \land \Diamond \varphi_n) \ldots))$$

$$\varphi_i = p_1 \land \ldots \land p_{i-1} \land \neg p_i \land p_{i+1} \land \ldots \land p_n$$

(a similar criterion for multi-modal logic).

The semantic condition for $\alpha_n$ is:

$$\neg \exists x_1, \ldots, x_n (x_1 R \ldots Rx_n \land \text{all } x_i \text{ are different})$$

Remember, however, that tabularity in (N)Ext$K$ is undecidable (Chagrov 1996).

Local tabularity in NExt$K$ is also undecidable (Chagrov 2002).

Chagrov’s conjecture (1994): $K + \alpha_n$ is locally tabular.

Shehtman proved this conjecture in 2014.
Post completeness

A logic $L$ is **Post complete** in $\text{Ext} K$ is $L$ if consistent and has no proper consistent extensions in $\text{Ext} K$

**Chagrov 1985:** $L$ is **generally Post complete** if $L$ is consistent and has no consistent extensions closed under the rules admissible in $L$

For every generally Post complete modal logic $L$, $L$ is Post complete in $\text{Ext} K$ iff $L$ is structurally complete

- there is a continuum of generally Post complete logics in $\text{NExt} K 4$
- there is a continuum of Post complete logics in $\text{Ext} K 4$

A logic is **antitabular** if it is consistent but does not have finite models
(a consistent logic is antitabular iff all its Post complete extensions are not tabular)

If $L \supseteq K 4$ has infinitely many Post complete extensions then it also has an antitabular extension
Other results

**Chagrov 1992:** There exists a continuum of maximal intermediate propositional logics with the disjunction property.
(Maksimova 1984: there exist infinitely many such logics. The only known explicit example is Medvedev logic.)

**Chagrov 2015:** There exists a normal modal logic $L$ with the FMP and a variable-free formula $\varphi$ such that $L + \varphi$ lacks the FMP.