Propositional Dynamic Logic With Belnapian Truth Values

Igor Sedlár

Institute of Computer Science, Czech Academy of Sciences, Prague

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Overview

BPDL, a four-valued paraconsistent version of propositional dynamic logic **PDL**

1. Motivation
2. Belnapian truth values
3. **BPDL** and what it can do
4. Properties of **BPDL**

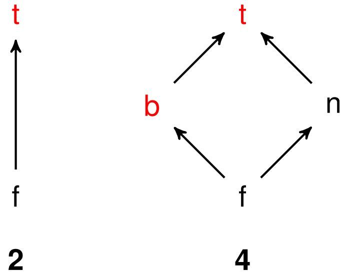
Motivation

Motivation

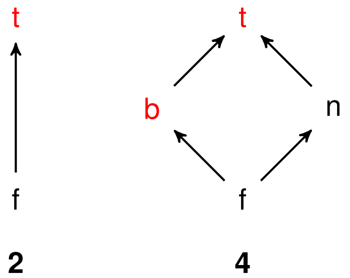
- **PDL** (Fischer and Ladner, 1979) is a (deductive) verification formalism used to prove correctness of programs, relations among programs etc.
- **PDL** models program states as complete and consistent possible worlds
- Programs understood more generally (e. g. database queries and transformations; algorithmic transformations of bodies of information) go beyond this; they require incomplete and inconsistent states
- Belnap (1977a, 1977b) and Dunn (1976) introduce such states
- We outline **BPDL**, a version of **PDL** built on an extension of the Belnap–Dunn logic studied by Odintsov and Wansing (2010)

Belnapian states

Classical and Belnapian states



Classical and Belnapian states



$$\bot^{\mathbf{L}} = \mathrm{f}$$

$$\sim^{\mathbf{L}} e = \begin{cases} \mathrm{f} & \text{if } e = \mathrm{t} \\ \mathrm{t} & \text{if } e = \mathrm{f} \\ e & \text{otherwise} \end{cases}$$

$$e \wedge^{\mathbf{L}} e' = inf\{e, e'\}$$

$$e \vee^{\mathbf{L}} e' = sup\{e, e'\}$$

$$e \to^{\mathbf{L}} e' = \begin{cases} e' & \text{if } e \in \mathcal{D}(\mathbf{L}) \\ \mathrm{t} & \text{otherwise} \end{cases}$$

$$\neg^{\mathbf{L}} e = e \to^{\mathbf{L}} \bot^{\mathbf{L}}$$

BK (Odintsov and Wansing, 2010)

Kripke L-models and BK

- $M = \langle S, R, v^{\mathbf{L}} \rangle$; $v^{\mathbf{L}} : (FRM \times W) \rightarrow \mathbf{L}$ (respects $\circ^{\mathbf{L}}$ for $\circ \in \{\bot, \sim, \wedge, \vee, \rightarrow\}$)
- $v^{\mathbf{L}}(\Box\phi, w) = inf\{v^{\mathbf{L}}(\phi, w') \mid Rww'\}$
- $v^{\mathbf{L}}(\Diamond\phi, w) = sup\{v^{\mathbf{L}}(\phi, w') \mid Rww'\}$
- $\Gamma \models^{\mathbf{L}} \phi$ iff $inf\{v^{\mathbf{L}}(\psi, w) \mid \psi \in \Gamma\} \in \mathcal{D}(\mathbf{L})$ only if $v^{\mathbf{L}}(\phi, w) \in \mathcal{D}(\mathbf{L})$ for all (M, w).
- **K** if $\mathbf{L} = \mathbf{2}$; **BK** if $\mathbf{L} = \mathbf{4}$

BK (Odintsov and Wansing, 2010)

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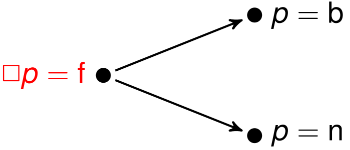
Example 1



BK (Odintsov and Wansing, 2010)

Theorem 2

The sound and complete axiomatization of **BK** *is*

1. **CL** *in* $\{AF, \bot, \to, \wedge, \vee\}$*;*
2. *Strong negation axioms:*

$$\sim\sim\phi \leftrightarrow \phi,\ \sim(\phi \wedge \psi) \leftrightarrow (\sim\phi \vee \sim\psi),\ \sim(\phi \vee \psi) \leftrightarrow (\sim\phi \wedge \sim\psi),$$
$$\sim(\phi \to \psi) \leftrightarrow (\phi \wedge \sim\psi),\ \top \leftrightarrow \sim\bot;$$

3. *The* **K** *axiom* $\Box(\phi \to \psi) \to (\Box\phi \to \Box\psi)$ *and the Necessitation rule* $\phi/\Box\phi$*;*
4. *Modal interaction principles:*

$$\neg\Box\phi \leftrightarrow \Diamond\neg\phi,\ \neg\Diamond\phi \leftrightarrow \Box\neg\phi,$$
$$\sim\Box\phi \leftrightarrow \Diamond\sim\phi,\ \Box\phi \leftrightarrow \sim\Diamond\sim\phi,$$
$$\sim\Diamond\phi \leftrightarrow \Box\sim\phi,\ \Diamond\phi \leftrightarrow \sim\Box\sim\phi.$$

Belnapian **PDL**

BPDL

Language

$(ACT)\quad \alpha ::= a \in ACT_0 \mid \alpha;\alpha \mid \alpha \cup \alpha \mid \alpha^* \mid \phi?$
$(FRM)\quad \phi ::= p \in FRM_0 \mid \bot \mid {\sim}\phi \mid \phi \wedge \phi \mid \phi \vee \phi \mid \phi \to \phi \mid [\alpha]\phi \mid \langle\alpha\rangle\phi$

Semantics

$\mathcal{M} = \langle S, R, v^{\mathbf{4}} \rangle$ where $R : ACT \mapsto \mathcal{P}(S^2)$ and $v^{\mathbf{4}}$ is as in **BK**-models (for all $\alpha \in ACT$). Moreover:

1. $R(\alpha;\beta) = R(\alpha) \circ R(\beta)$
2. $R(\alpha \cup \beta) = R(\alpha) \cup R(\beta)$
3. $R(\alpha^*) = R(\alpha)^*$
4. $R(\phi?) = \{\langle x, x \rangle \mid v^{\mathbf{4}}(\phi, x) \in \mathcal{D}(\mathbf{4})\}$

Examples I

'Not false'

$\neg{\sim}p$ means that p is not false. As a result, the four Belnapian truth values are expressible as

- $p \wedge \neg{\sim}p$ (t, 'true and not false')
- $p \wedge {\sim}p$ (b, 'true and false')
- $\neg p \wedge {\sim}p$ (f, 'false and not true')
- $\neg p \wedge \neg{\sim}p$ (n, 'neither true nor false')

Default rules

Every default rule d of the form $\frac{p : q}{r}$ can be represented by an atomic program a_d satisfying $(p \wedge \neg{\sim}q) \rightarrow [a_d]r$

Examples II

Inconsistency handling strategies

- **If-then-else** 'If there is inconsistent information about p, then do a_p (else b_p)', 'if there is inconsistent information about q, then do a_q (else b_q)': $(p \wedge {\sim}p)?; a_p \cup \neg(p \wedge {\sim}p)?; b_p$ and $(q \wedge {\sim}q)?; a_q \cup \neg(q \wedge {\sim}q)?; b_q$
- **While** 'While there is inconsistent information about p, do a_p': $((p \wedge {\sim}p)?; a_p)^*; \neg(p \wedge {\sim}p)?$

Adding and removing information

Actions of adding or removing p to/from a database can be represented by atomic programs satisfying $[a^{+p}]p$ and $[a^{-p}]\neg p$.

Properties of **BPDL**

BPDL and PDL

Theorem 3

The **PDL** *axioms*

$$
\begin{aligned}
[\alpha \cup \beta]\phi &\leftrightarrow ([\alpha]\phi \wedge [\beta]\phi) \\
[\alpha ; \beta]\phi &\leftrightarrow [\alpha][\beta]\phi \\
[\psi ?]\phi &\leftrightarrow (\psi \rightarrow \phi) \\
[\alpha^*]\phi &\leftrightarrow (\phi \wedge [\alpha][\alpha^*]\phi) \\
[\alpha^*]\phi &\leftarrow (\phi \wedge [\alpha^*](\phi \rightarrow [\alpha]\phi))
\end{aligned}
$$

are valid in **BPDL** *(and so are their 'diamond versions').*

Theorem 4

BPDL *is not compact.*

Deduction theorem and decidability

Theorem 5

For finite Γ *with all atomic programs in* $\{a_1, \ldots, a_n\}$*:*

1. $\Gamma \models \phi$ *iff* $\models \bigwedge \Gamma \rightarrow \phi$
2. $\Gamma \models^{g} \phi$ *iff* $\models [(a_1 \cup \ldots \cup a_n)^*] \bigwedge \Gamma \rightarrow \phi$

Deduction theorem and decidability

Theorem 5

For finite Γ *with all atomic programs in* $\{a_1, \ldots, a_n\}$*:*

1. $\Gamma \models \phi$ *iff* $\models \bigwedge \Gamma \to \phi$
2. $\Gamma \models^g \phi$ *iff* $\models [(a_1 \cup \ldots \cup a_n)^*] \bigwedge \Gamma \to \phi$

Theorem 6

$\models \phi$ is decidable *(but* $\Gamma \models^g \phi$ *for infinite* Γ *is (highly) undecidable).*

Proof.

Standard filtration argument. The equivalence classes in the filtration are defined to coincide on all $\phi, \sim\phi$ where $\phi \in FL(\psi)$. $\square$

Completeness

Theorem 7

A sound and weakly complete axiomatisation of **BPDL** *extends the (ACT-dimensional) axiomatisation of* **BK** *by the standard* **PDL** *axioms and their diamond versions.*

Proof.

Filtration of the canonical structure. □

Summary and future work

In conclusion

Summary

- **PDL** with non-standard states is relevant to formal verification of ‘information-modifying’ programs (such as, e.g., database transformations)
- **BPDL** is a well-behaved decidable formalism that can be used

Future work

- Complexity of **BPDL**
- Other non-classical versions of **PDL**, for example: substructural **PDL**, fuzzy **PDL**
- Extensions to other program logics such as Dynamic Logic **DL** and Process Logic **PL**

Thank you!

References

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