Semi-provability predicates and extensions of **GL**

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Arithmetical Completeness of GL

$\mathrm{Prv}_{\mathrm{PA}}(x)$ is the Gödel provability predicate for PA.

$(\cdot)^\star \colon \mathbf{L}_\Box \to \mathbf{ArithSen}$ is an interpretation with respect to $\mathrm{Prv}_{\mathrm{PA}}(x)$ if

- $(\Box\varphi)^\star$ is equal to $\mathrm{Prv}_{\mathrm{PA}}(\ulcorner \varphi^\star \urcorner)$;
- $(\cdot)^\star$ commutes with all non-modal connective;
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Theorem (Solovay 1976)

For every modal formula φ the following conditions are equivalent:

- $\mathrm{GL} \vdash \varphi$;
- *for every interpretation $(\cdot)^\star$ with respect to $\mathrm{Prv}_{\mathrm{PA}}(x)$ we have $\mathrm{PA} \vdash \varphi^\star$.*

Semi-provability predicates

Suppose T interpretes PA.

Hilbert-Bernays-Löb derivability conditions for $\mathrm{SPrv}(x)$:

- $\mathrm{T} \vdash \varphi \;\Rightarrow\; \mathrm{T} \vdash \mathrm{SPrv}(\ulcorner \varphi \urcorner)$;
- $\mathrm{T} \vdash \mathrm{SPrv}(\ulcorner \varphi \to \psi \urcorner) \to (\mathrm{SPrv}(\ulcorner \varphi \urcorner) \to \mathrm{SPrv}(\ulcorner \psi \urcorner))$;
- $\mathrm{T} \vdash \mathrm{SPrv}(\ulcorner \varphi \urcorner) \to \mathrm{SPrv}(\mathrm{SPrv}(\ulcorner \varphi \urcorner))$.

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In this case we call $\mathsf{SPrv}(x)$ semi-provability predicate.

$\mathcal{L}(\mathsf{SPrv})$ is the set of all modal formulas φ such that for every interpetation $(\cdot)^\star$ with respect to SPrv we have $\mathsf{T} \vdash \mathsf{SPrv}(\ulcorner \varphi^\star \urcorner)$.

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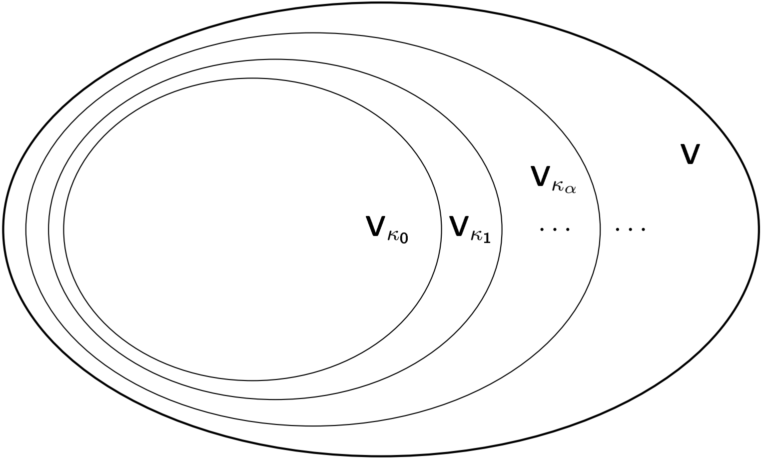
Solovay example:
Suppose $\mathsf{Univ}(x)$ is a semi-provability predicate in ZFC such that $\mathsf{Univ}(\ulcorner\varphi\urcorner)$ means that φ is true in all the models $(\mathbf{V}_\kappa, \in)$, where κ is inaccessible.

If we take ZFC + "there exists infinitely many inaccessible cardinals" as our metatheory then:

$$\mathcal{L}(\mathsf{Univ}) = \mathsf{GL.3}.$$

Explanation of Solovay Example

κ_α is the α-th inaccessible.



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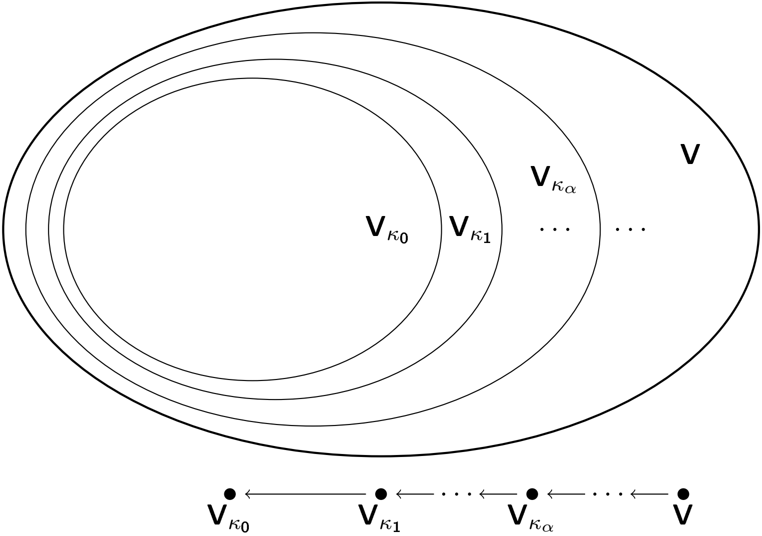


Kripke Complete Semi-Provability Predicates

Theorem

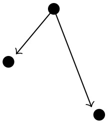
Suppose L *is a Kripke complete recursively axiomatizable extension of* GL *and* ZFC *proves that* L *is Kripke complete. Then there exists a* ZFC *semi-provability predicate* $\mathrm{SPrv}_{\mathrm{L}}(x)$ *such that* $\mathcal{L}(\mathrm{SPrv}_{\mathrm{L}}) = \mathrm{L}$.

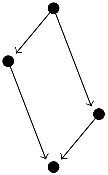
(theorem is provable in ZFC + *"there exists a proper class of inaccessible cardinals")*

Embedding Kripke Frames into Model Accessibility

L-Kripke frames

⋮





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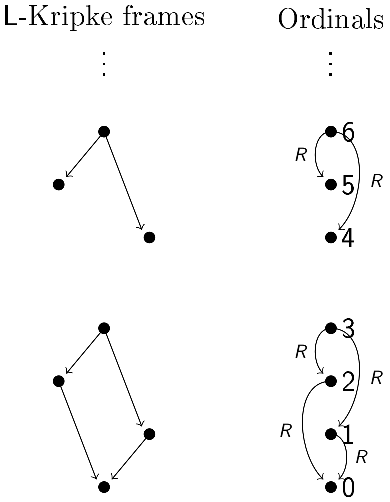

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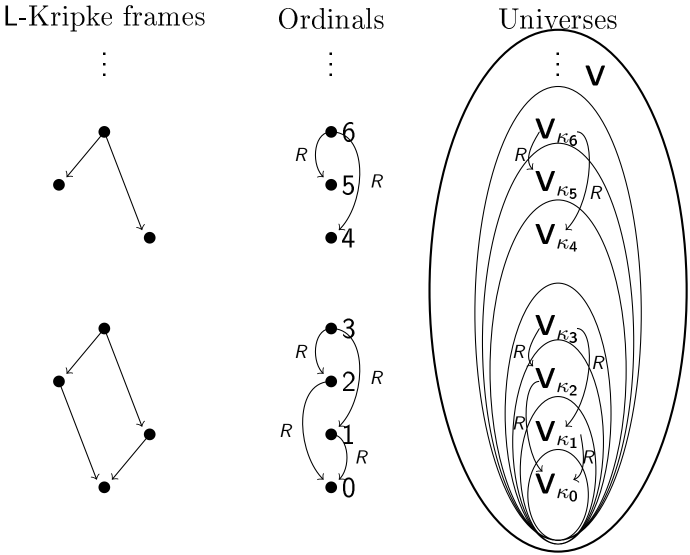

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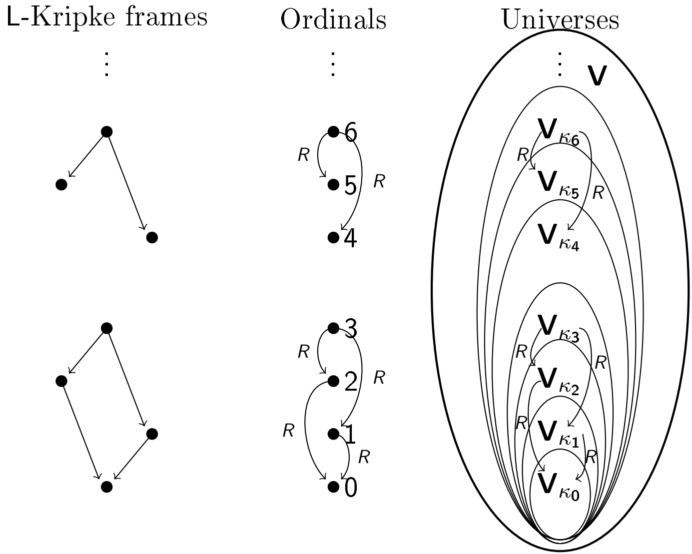

$$\mathbf{V}_{\kappa_\alpha} \models \mathsf{SPrv}_{\mathsf{L}}(\ulcorner\varphi\urcorner) \overset{\text{def}}{\iff} \forall\beta(\alpha\ R\ \beta \Rightarrow \mathbf{V}_{\kappa_\beta} \models \varphi).$$

The Case of Arithmetical Theories

Theorem

Suppose T *is a recursively axiomatizable* Σ_2*-sound extension of* PA *and* L *is a consistent enumerable normal extension of* GL*. If* T *proves that* L *have FMP then there is a semi-provability predicate* $\mathsf{SPrv}_{\mathsf{L}}(x)$ *in* T *such that* $\mathcal{L}(\mathsf{SPrv}_{\mathsf{L}}) = \mathsf{L}$.

Thank You!