The Structure of the Lattice of Normal Modal Logics with Cyclic Axioms

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Ingredients of this research

(1) The class *Irr* of all irreflexive (general) frames which consist of only irreflexive points ($\circ$)

(2) Cyclic axioms ($\mathrm{Cycl}(n) := p \to \Box^n \Diamond p$ for $n \geq 0$)

(3) A criterion for a modal algebra to be s.i.

(4) Well-known facts on the logic $\mathbf{L}(\circ)$

(5) Splitting of a lattice of normal modal logics

Two one-point frames and their algebras

The frame of one reflexive point $\bullet \Longrightarrow$ algebra $\mathbf{2}^r$

The frame of one irreflexive point $\circ \Longrightarrow$ algebra $\mathbf{2}^i$

Irreflexive frames

$\mathcal{F} := \langle W, R, P \rangle$: a (general) frame

(1) A point $a \in W$ is irreflexive if aRa does not hold.

(2) A frame $\mathcal{F}$ is irreflexive if every point in $\mathcal{F}$ is irreflexive.

(3) Every irreflexive point is drawn by a circle ($\circ$).

(4) Irr is the class of all irreflexive (general) frames.

Cyclic axioms

$\mathrm{Cycl}(n) := p \to \Box^n \Diamond p$ for $n \geq 0$

For a frame $\mathcal{F}$,

$\mathcal{F} \models \mathrm{Cycl}(n)$

$\Leftrightarrow \mathcal{F} \models\!\!\!\equiv \forall x_0, x_1, \ldots, x_n \left({}_{x_0}R_{x_1}R_{x_2} \cdots R_{x_n} \Rightarrow {}_{x_n}R_{x_0} \right)$

$\Leftrightarrow \mathcal{F}$ is *n-cyclic*.

== Note ==

$\mathrm{Cycl}(0) = \mathbf{T}$, $\mathrm{Cycl}(1) = \mathbf{B}$.

Subdirectly Irreducible Modal algebra

For a non-trivial modal algebra $\mathfrak{A} = \langle A, \cap, \cup, -, \Box, 0, 1 \rangle$,

$\mathfrak{A}$ is subdirectly irreducible

$\Leftrightarrow \exists d (\neq 1) \in A, \forall x (\neq 1) \in A, \exists n \in \omega$ s.t.

$$x \cap \Box x \cap \Box^2 x \cap \cdots \cap \Box^n x \leq d$$

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Theorem (D. Makinson (1971))

For any consistent modal logic $\mathbf{L}$*, either* $\mathbf{L} \subseteq \mathbf{L}(\bullet)$ *or* $\mathbf{L} \subseteq \mathbf{L}(\circ)$ *holds.* □

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Proposition

$(\mathbf{KD}, \mathbf{L}(\circ))$ *is a splitting pair of the lattice* $\mathrm{NExt}(\mathbf{K})$*.* $\square$

$*$ $\mathbf{D} := \Diamond\top$

$\mathcal{F} \models \mathbf{D} \;\Leftrightarrow\; \mathcal{F} \models \forall x \exists y (xRy)$ (seriality).

Splitting

= Definition =

$\mathcal{L} := \langle L, \wedge, \vee, 0, 1 \rangle$:
a complete lattice
$a \in L$ splits $\mathcal{L}$ if there exists $b \in L$ s.t. for any $x \in L$, either $x \leq a$ or $b \leq x$, but not both.
Such a pair (b, a) is called a splitting pair of $\mathcal{L}$.

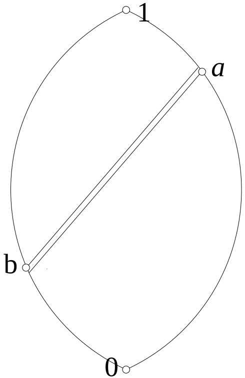

Figure: A splitting of a complete lattice $\mathcal{L}$

Our original question (1)

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(Case 2) If $\circ$ is isomorphic to a generated subframe of some points in $\mathcal{F}$, then $\mathbf{L}(\mathcal{F}) \subseteq \mathbf{L}(\circ)$.

(Case 3) If $\circ$ is contained as a disjoint component in $\mathcal{F}$, then $\mathbf{L}(\mathcal{F}) \subseteq \mathbf{L}(\circ)$.

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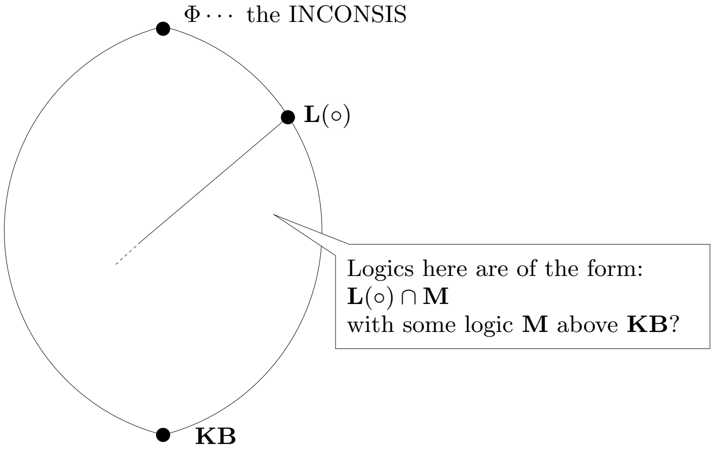

Figure: NExt(KB)

Fact

Let $\mathfrak{A}$ be a non-trivial s.i. modal algebra. Suppose $\Box 0 = 1$ in $\mathfrak{A}$. Then for any $x \in A$, if $x \neq 1$, then $x = 0$.

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Take any $x(\neq 1) \in A$. Because $\mathfrak{A}$ is s.i., there is $d(\neq 1) \in A$, for this x, there is a number n s.t. $x \cap \Box x \cap \Box^2 x \cap \cdots \cap \Box^n x \leq d$ holds. Thus $x \leq d$.

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Fact

$\mathbf{2}^i$ is the only s.i. algebra which satisfies $\Box 0 = 1$. □

Situation over KB

Theorem

Let $\mathfrak{A}$ *be a non-trivial s.i. algebra for Cycl*$(1)=\mathbf{B}$. *Suppose* $\Diamond 1 \neq 1$ *in* $\mathfrak{A}$. *Then* $\Box 0 = 1$.

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This means that $(\mathbf{KDB}, \mathbf{L}(\circ))$ is a splitting pair over $\mathbf{KB}$!

Lattice-Mapping

Define maps σ and τ in the following:

$$\sigma : \textsc{NExt}(\mathbf{KDB}) \to [\mathbf{KB}, \mathbf{L}(\circ)]$$

$$\sigma(\mathbf{L}) := \mathbf{L} \cap \mathbf{L}(\circ)$$

$$\tau : [\mathbf{KB}, \mathbf{L}(\circ)] \to \textsc{NExt}(\mathbf{KDB})$$

$$\tau(\mathbf{M}) := \mathbf{M} \vee \mathbf{KDB}$$

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Show that σ is an isomorphism!

Lattice-homomorphism

Lemma

σ is a lattice-homomorphism.

Proof: For logics $\mathbf{L}_1, \mathbf{L}_2 \in \mathrm{NE}\textsc{xt}(\mathbf{KDB})$,

$$\begin{aligned} \sigma(\mathbf{L}_1 \cap \mathbf{L}_2) &= \mathbf{L}_1 \cap \mathbf{L}_2 \cap \mathbf{L}(\circ) \\ &= \mathbf{L}_1 \cap \mathbf{L}(\circ) \cap \mathbf{L}_2 \cap \mathbf{L}(\circ) \\ &= \sigma(\mathbf{L}_1) \cap \sigma(\mathbf{L}_2) \end{aligned}$$

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$\square$

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$\mathbf{KB} = \mathbf{KDB} \cap \mathbf{L}(\circ)$

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Suppose $\varphi \notin \mathbf{KB}$ for some formula φ. Then there is a frame $\mathcal{F}$ for $\mathbf{B}$, a valuation V on $\mathcal{F}$ and a point a in $\mathcal{F}$ s.t. $\langle \mathcal{F}, V \rangle \not\models_a \varphi$.

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Hence $\varphi \notin \mathbf{KDB} \cap \mathbf{L}(\circ)$, and so, $\mathbf{KB} \supseteq \mathbf{KDB} \cap \mathbf{L}(\circ)$ □

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σ is onto.

Proof: For any $\mathbf{M} \in [\mathbf{KB}, \mathbf{L}(\circ)]$,

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Thus $\langle \mathcal{F}, V \rangle \not\models_a \varphi \vee \Box\Box\varphi$.
Hence $\varphi \vee \Box\Box\varphi \notin \mathbf{L}_2 \cap \mathbf{L}(\circ)$.

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$\Box\bot \to \Box\Box\varphi \in \mathbf{K} \subseteq \mathbf{L}(\circ)$. So, $\Box\Box\varphi \in \mathbf{L}(\circ)$.

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Therefore, $\varphi \vee \Box\Box\varphi \in \mathbf{L}_1 \cap \mathbf{L}(\circ)$.

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Therefore, $\varphi \vee \Box\Box\varphi \in \mathbf{L}_1 \cap \mathbf{L}(\circ)$.
Thus, $\sigma(\mathbf{L}_1) = \mathbf{L}_1 \cap \mathbf{L}(\circ) \not\subseteq= \mathbf{L}_2 \cap \mathbf{L}(\circ) = \sigma(\mathbf{L}_2)$.
This means that σ is one to one. $\Box$

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Theorem

$[\mathbf{KB}, \mathbf{L}(\circ)]$ *is isomorphic to* NEXT(**KDB**). □

A Result on NExt(KB)

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The "onto" lemma says that for any $\mathbf{M} \in [\mathbf{KB}, \mathbf{L}(\circ)]$, there exists $\mathbf{L} \in$ NExt($\mathbf{KDB}$), s.t. $\mathbf{M} = \mathbf{L} \cap \mathbf{L}(\circ)$.

A Result on $\mathrm{NExt}(\mathbf{KB})$

We have shown the following:

Theorem

$[\mathbf{KB}, \mathbf{L}(\circ)]$ *is isomorphic to* $\mathrm{NExt}(\mathbf{KDB})$. □

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This answers the original question.

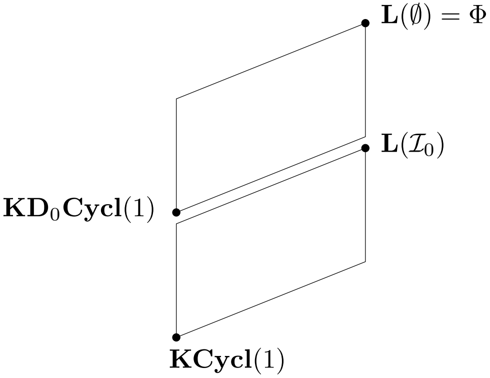

Figure: The structure of $\mathrm{NE{\scriptsize XT}}(\mathbf{KCycl}(1))$

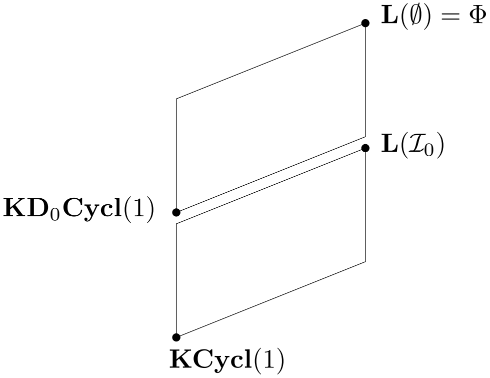

Figure: The structure of $\mathrm{NE}\textsc{xt}(\mathbf{KCycl}(1))$

$\mathrm{NE}\textsc{xt}(\mathbf{KB})$ looks like a two-story building!

Generalization to NExt(**KCycl**(2))

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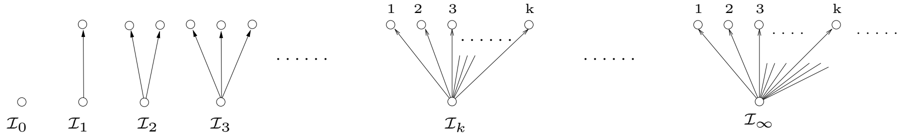

Figure: Irreflexive frames for **KCycl**(2)

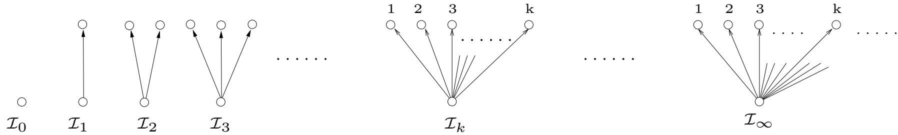

Figure: Irreflexive frames for **KCycl**(2)

Proposition

$\mathbf{L}(\mathcal{I}_0) \supsetneq \mathbf{L}(\mathcal{I}_1) \supsetneq \mathbf{L}(\mathcal{I}_2) \supsetneq \cdots \supsetneq \mathbf{L}(\mathcal{I}_\infty)$.

Serial axioms

$\mathbf{D}_n := \Box^n \Diamond \top$ for $n \geq 0$

For a frame $\mathcal{F}$,

$\mathcal{F} \models \mathbf{D}_n$

$\Leftrightarrow \mathcal{F} \models\!\!\models \forall x_0, x_1, \ldots, x_n \big({}_{x_0}R_{x_1}R_{x_2} \cdots R_{x_n} \Rightarrow (\exists y \text{ s.t. } {}_{x_n}R_y) \big)$

$\Leftrightarrow \mathcal{F}$ is n-serial.

== Note ==

$\mathbf{D}_0 = \mathbf{D}$.

Theorem

For any $k \geq 1$, $(\mathbf{KD_0Cycl}(2) \cap \mathbf{L}(\mathcal{I}_{k-1}), \mathbf{L}(\mathcal{I}_k))$ is a splitting pair in $\mathrm{NExt}(\mathbf{KCycl}(2))$. □

Isomorphism Theorem in $\mathrm{NExt}(\mathbf{KCycl}(2))$

Theorem

$\mathrm{NExt}(\mathbf{KD}_0\mathbf{Cycl}(2))$ *is isomorphic to the interval* $[\mathbf{KD}_0\mathbf{Cycl}(2) \cap \mathbf{L}(\mathcal{I}_k), \mathbf{L}(\mathcal{I}_k)]$ *for each* $k \geq 0$. □

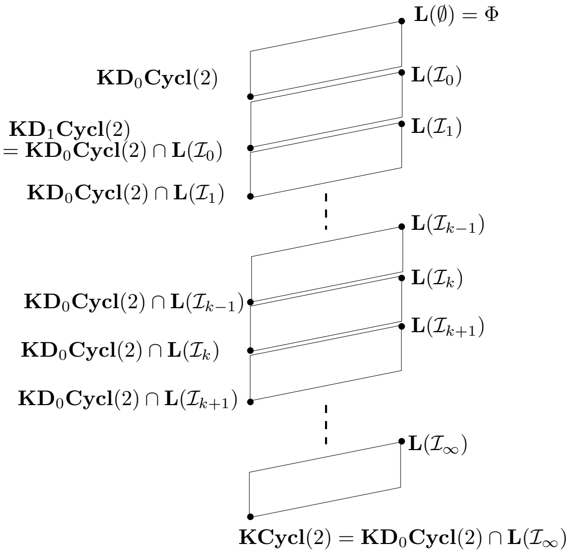

Figure: The structure of NExt($\mathbf{KCycl}(2)$)

Facts on $\mathrm{NExt}(\mathbf{KCycl}(1))$ and $\mathrm{NExt}(\mathbf{KCycl}(2))$

(1) There is an essential lattice structure of logics at the top-most part of $\mathrm{NExt}(\mathbf{KCycl}(1))$ and $\mathrm{NExt}(\mathbf{KCycl}(2))$.

(2) That rest part has a repeated structure of the essential part.

Conjecture on $\mathrm{NExt}(\mathbf{KCycl}(n))$ for $n \geq 1$

$\mathcal{B}_n$: an essential lattice sturucture of logics in $\mathrm{NExt}(\mathbf{KCycl}(n))$

$$\mathcal{I}rr_n := \{\mathbf{L}(\mathcal{C}) \in \mathrm{NExt}(\mathbf{KCycl}(n)) \mid \mathcal{C} \text{ is a class of some irreflexive frames}\}$$

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$\mathcal{B}_n$: an essential lattice sturucture of logics in $\mathrm{NExt}(\mathbf{KCycl}(n))$

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$$\mathrm{NExt}(\mathbf{KCycl}(n)) \cong \mathcal{B}_n \times \mathcal{I}rr_n$$

for every $n \geq 1$?

A SPLITING OVER $\mathbf{KCycl}(n)$: $(n \geq 1)$

Theorem

Let $\mathfrak{A}$ be a non-trivial s.i. algebra for $Cycl(n) = p \to \Box^n \Diamond p$. Suppose $\Box^{n-1}\Diamond 1 \neq 1$ in $\mathfrak{A}$. Then $\Box^n 0 = 1$. □

A splitting over $\mathbf{KCycl}(n)$: $(n \geq 1)$

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Theorem

$(\mathbf{KCycl}(n)\mathbf{D}_{n-1}, \mathbf{L}(\mathcal{C}h_n))$ *is a splitting pair in* $\mathrm{NExt}(\mathbf{KCycl}(n))$. □

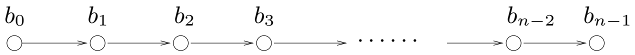

Figure: Frames $\mathcal{C}h_n$