Axiomatizing a Real-Valued Modal Logic

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Łukasiewicz Modal Logics

Hansoul and Teheux (2013) consider a **Łukasiewicz modal logic** with

- standard "crisp" Kripke frames
- connectives defined on the real unit interval $[0, 1]$

$$x \to y = \min(1, 1 - x + y) \qquad \neg x = 1 - x$$
$$x \oplus y = \min(1, x + y) \qquad x \odot y = \max(0, x + y - 1)$$

- $\Box$ and $\Diamond$ interpreted as infima and suprema of accessible values.

Łukasiewicz multi-modal logics can also be viewed as fragments of **continuous logic** and have been studied as **fuzzy description logics**.

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An Axiomatization Problem

Hansoul and Teheux (2013) obtain an axiomatization of Łukasiewicz modal logic by extending an axiomatization of Łukasiewicz logic with

$$\Box(\varphi \to \psi) \to (\Box\varphi \to \Box\psi)$$
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$$\frac{\varphi}{\Box\varphi}$$

and a rule with infinitely many premises

$$\frac{\varphi \oplus \varphi \quad \varphi \oplus \varphi^2 \quad \varphi \oplus \varphi^3 \quad \ldots}{\varphi}$$

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Towards a Solution...

We provide a finitary axiomatizion of a **real-valued modal logic** that extends the multiplicative fragment of Abelian logic.

Extending this system with the additive (lattice) connectives would provide the basis for a finitary axiomatizion for Łukasiewicz modal logic.

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The Multiplicative Fragment of Abelian Logic

The **multiplicative fragment of Abelian logic** is axiomatized by

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\begin{array}{ll}
\text{(B)} & (\varphi \to \psi) \to ((\psi \to \chi) \to (\varphi \to \chi)) \\
\text{(C)} & (\varphi \to (\psi \to \chi)) \to (\psi \to (\varphi \to \chi)) \\
\text{(I)} & \varphi \to \varphi \\
\text{(A)} & ((\varphi \to \psi) \to \psi) \to \varphi
\end{array}
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$$\frac{\varphi \quad \varphi \to \psi}{\psi}\ (\text{mp})$$

and is complete with respect to the logical matrix

$$\langle \mathbb{R}, \mathbb{R}_{\geq 0}, \{\to\} \rangle \quad \text{where} \quad x \to y = y - x.$$

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A Modal Language

We define further connectives (for a fixed variable p_0)

$$\begin{aligned} \overline{0} &:= p_0 \to p_0 \\ \neg\varphi &:= \varphi \to \overline{0} \\ \varphi + \psi &:= \neg\varphi \to \psi. \end{aligned}$$

For our modal language, we add a unary connective $\Box$, and define

$$\Diamond\varphi := \neg\Box\neg\varphi.$$

The set of formulas Fm for this language is defined inductively as usual over a countably infinite set of variables Var.

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Frames and Models

A **frame** $\mathfrak{F} = \langle W, R \rangle$ consists of

- a non-empty set of **worlds** W
- an **accessibility relation** $R \subseteq W \times W$.

$\mathfrak{F}$ is called **serial** if for all $x \in W$, there exists $y \in W$ such that Rxy.

Models

A $\mathrm{K}(\mathbb{R})$-**model** $\langle W, R, V \rangle$ consists of

- a **serial frame** $\langle W, R \rangle$
- an **evaluation map** $V \colon \mathrm{Var} \times W \to [-r, r]$ for some $r > 0$.

The evaluation map is extended to $V \colon \mathrm{Fm} \times W \to \mathbb{R}$ by

$$\begin{aligned} V(\varphi \to \psi, x) &= V(\psi, x) - V(\varphi, x) \\ V(\Box\varphi, x) &= \inf\{V(\varphi, y) : Rxy\}. \end{aligned}$$

It follows also that

$$\begin{aligned} V(\overline{0}, x) &= 0 & V(\varphi + \psi, x) &= V(\varphi, x) + V(\psi, x) \\ V(\neg\varphi, x) &= -V(\varphi, x) & V(\Diamond\varphi, x) &= \sup\{V(\varphi, y) : Rxy\}. \end{aligned}$$

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Validity

A formula φ is

- **valid** in a $\mathrm{K}(\mathbb{R})$-model $\langle W, R, V\rangle$ if $V(\varphi, x) \geq 0$ for all $x \in W$
- $\mathrm{K}(\mathbb{R})$-**valid** if it is valid in all $\mathrm{K}(\mathbb{R})$-models.

Lemma

The following are equivalent for any formula φ:

(1) φ *is* $\mathrm{K}(\mathbb{R})$*-valid.*

(2) φ *is valid in all finite* $\mathrm{K}(\mathbb{R})$*-models.*

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The Axiom System $\mathrm{K}(\mathbb{R})$

(B)	$(\varphi \to \psi) \to ((\psi \to \chi) \to (\varphi \to \chi))$
(C)	$(\varphi \to (\psi \to \chi)) \to (\psi \to (\varphi \to \chi))$
(I)	$\varphi \to \varphi$
(A)	$((\varphi \to \psi) \to \psi) \to \varphi$
(K)	$\Box(\varphi \to \psi) \to (\Box\varphi \to \Box\psi)$
(P)	$\Box(\varphi + \cdots + \varphi) \to (\Box\varphi + \cdots + \Box\varphi)$

$$\frac{\varphi \quad \varphi \to \psi}{\psi}\ \text{(mp)} \qquad \frac{\varphi}{\Box\varphi}\ \text{(nec)} \qquad \frac{\varphi + \cdots + \varphi}{\varphi}\ \text{(con)}$$

The Sequent Calculus $\mathrm{GK}(\mathbb{R})$

$$\frac{}{\Delta \Rightarrow \Delta}\ (\textsc{id}) \qquad \frac{\Gamma, \varphi \Rightarrow \Delta \qquad \Pi \Rightarrow \varphi, \Sigma}{\Gamma, \Pi \Rightarrow \Sigma, \Delta}\ (\textsc{cut})$$

$$\frac{\Gamma \Rightarrow \Delta \quad \Pi \Rightarrow \Sigma}{\Gamma, \Pi \Rightarrow \Sigma, \Delta}\ (\textsc{mix}) \qquad \frac{\Gamma, \ldots, \Gamma \Rightarrow \Delta, \ldots, \Delta}{\Gamma \Rightarrow \Delta}\ (\textsc{sc})$$

$$\frac{\Gamma, \psi \Rightarrow \varphi, \Delta}{\Gamma, \varphi \to \psi \Rightarrow \Delta}\ (\to\Rightarrow) \qquad \frac{\Gamma, \varphi \Rightarrow \psi, \Delta}{\Gamma \Rightarrow \varphi \to \psi, \Delta}\ (\Rightarrow\to)$$

$$\frac{\Gamma \Rightarrow \varphi, \ldots, \varphi}{\Box\Gamma \Rightarrow \Box\varphi, \ldots, \Box\varphi}\ (\Box)$$

Equivalence of Proof Systems

We interpret sequents by

$$(\varphi_1, \ldots, \varphi_n \Rightarrow \psi_1, \ldots, \psi_m)^{\mathcal{I}} := (\varphi_1 + \cdots + \varphi_n) \to (\psi_1 + \ldots + \psi_m),$$

where $\varphi_1 + \cdots + \varphi_n := \overline{0}$ for $n = 0$.

Theorem

The following are equivalent:

(1) $\Gamma \Rightarrow \Delta$ *is derivable in* $\mathrm{GK}(\mathbb{R})$.

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The Main Result

Theorem

The following are equivalent for any formula φ*:*

(1) φ *is derivable in* $\mathrm{K}(\mathbb{R})$.

(2) φ *is* $\mathrm{K}(\mathbb{R})$*-valid.*

(3) $\Rightarrow \varphi$ *is derivable in* $\mathrm{GK}(\mathbb{R})$.

Proof Idea for (2) $\Rightarrow$ (3)

We prove by induction on the complexity of a sequent S that

$$S^{\mathcal{I}} \text{ is } \mathrm{K}(\mathbb{R})\text{-valid} \quad \Longrightarrow \quad S \text{ is derivable in } \mathrm{GK}(\mathbb{R}).$$

The base case where S contains no boxes is easy and the cases where S contains an implication follow using the invertibility of $(\to\Rightarrow)$ and $(\Rightarrow\to)$.

If S contains only boxed formulas and variables, then the multisets of variables on the left and right must coincide, and can be cancelled.

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Proof Idea for (2) $\Rightarrow$ (3) Continued

Suppose then that S is $\Box\Gamma \Rightarrow \Box\varphi_1, \ldots, \Box\varphi_n$. We apply the following $\mathrm{GK}(\mathbb{R})$-derivable rule for some $k > 0$ and $k\Gamma = \Gamma_0, \Gamma_1, \ldots, \Gamma_n$:

$$\frac{\Gamma_0 \Rightarrow \quad \Gamma_1 \Rightarrow k[\varphi_1] \quad \ldots \quad \Gamma_n \Rightarrow k[\varphi_n]}{\Box\Gamma \Rightarrow \Box\varphi_1, \ldots, \Box\varphi_n}$$

Using the $\mathrm{K}(\mathbb{R})$-validity of S, we generate (via labelled tableau rules) an inconsistent set of linear inequations. This inconsistency is witnessed by a linear combination of sequents where k is the coefficient of S. Eliminating variables we get the $\mathrm{K}(\mathbb{R})$-validity of $\Gamma_0 \Rightarrow$, $\Gamma_1 \Rightarrow k[\varphi_1], \ldots, \Gamma_n \Rightarrow k[\varphi_n]$.

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Complexity

Using our labelled tableau rules, we also obtain:

Theorem

Checking $\mathrm{K}(\mathbb{R})$*-validity of formulas is in EXPTIME.*

Concluding Remarks

There remain many issues to resolve:

- Can we add extend our axiomatization to an "Abelian modal logic" with lattice connectives? Do we obtain a Łukasiewicz modal logic?
- Can we develop useful algebraic semantics for these logics?
- Is the complexity of checking $\mathrm{K}(\mathbb{R})$-validity EXPTIME-complete? What is the complexity of validity in Łukasiewicz modal logic?

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