Neighbourhood products of pretransitive logics with S5

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Language and logics

$$\phi ::= p \mid \neg\phi \mid \phi \vee \phi \mid \Box_i \phi, \; i = 1, 2.$$

$\bot$, $\to$ and $\Diamond_i$ are expressible in the usual way.
Normal modal logic.
$\mathsf{K_n}$ denotes the minimal normal modal logic with n modalities and $\mathsf{K} = \mathsf{K_1}$.
$\mathsf{L_1}$ and $\mathsf{L_2}$ — two modal logics with one modality $\Box$ then the fusion of these logics is defined as

$$\mathsf{L_1} * \mathsf{L_2} = \mathsf{K_2} + \mathsf{L_1'} + \mathsf{L_2}';$$

where $\mathsf{L_i'}$ is the set of all formulas from $\mathsf{L_i}$ where in all formulas $\Box$ is replaced by $\Box_i$.

Topological and derivational semantics

Semantics on topological spaces can be built using closure operator cl where $cl(A)$ is the closure of A. The semantics defined like this:

$$V_{cl}(\Diamond\phi) = cl(V_{cl}(\phi))$$

Or using derivative operator d, where $d(A)$ is the set of all limit points of A. The semantics defined like this:

$$V_d(\Diamond\phi) = d(V_d(\phi))$$

	closure semantics	derivational semantics
all spaces	S4 (McKinsey & Tarski'1944)	wK4 (Esakia'1981)
$\mathbb{Q}$, Cantor space	S4	D4 (Shehtman'1990)
$\mathbb{R}$	S4	D4 + G_2 (Shehtman'2000)
$\mathbb{R}^n,\ n \geq 2$	S4	D4 + G_1 (Shehtman'1990)

wK4 = K + $\Diamond\Diamond p \to \Diamond p \vee p$
D4 = K + $\Diamond\Diamond p \to \Diamond p + \Diamond\top$

The product of Kripke frames

For two frames $F_1 = (W_1, R_1)$ and $F_2 = (W_2, R_2)$

$$F_1 \times F_2 = (W_1 \times W_2, R_1^*, R_2^*), \text{ where } (a_1,a_2)R_1^*(b_1,b_2) \Leftrightarrow a_1 R_1 b_1 \ \& \ a_2 = b_2$$
$$(a_1,a_2)R_2^*(b_1,b_2) \Leftrightarrow a_1 = b_1 \ \& \ a_2 R_2 b_2$$

For two logics L_1 and L_2

$$\mathsf{L}_1 \times \mathsf{L}_2 = Log(\{F_1 \times F_2 \mid F_1 \models \mathsf{L}_1 \ \& \ F_2 \models \mathsf{L}_2\})$$

(Shehtman, 1978)
For two classes of frames $\mathfrak{F}_1$ and $\mathfrak{F}_2$
$$Log(\{F_1 \times F_2 \mid F_1 \in \mathfrak{F}_1 \ \& \ F_2 \in \mathfrak{F}_2\}) \supseteq Log(\mathfrak{F}_1) * Log(\mathfrak{F}_2) + \Box_1\Box_2 p \leftrightarrow \Box_1\Box_2 p + \Diamond_1\Box_2 p \to \Box_2\Diamond_1 p.$$

$$\mathsf{K} \times \mathsf{K} = \mathsf{K} * \mathsf{K} + \Box_1\Box_2 p \leftrightarrow \Box_1\Box_2 p + \Diamond_1\Box_2 p \to \Box_2\Diamond_1 p$$
$$\mathsf{S4} \times \mathsf{S4} = \mathsf{S4} * \mathsf{S4} + \Box_1\Box_2 p \leftrightarrow \Box_1\Box_2 p + \Diamond_1\Box_2 p \to \Box_2\Diamond_1 p$$

$$\vdots$$

The product of topological spaces

(van Benthem et al, 2005)
For two topological space $\mathfrak{X}_1 = (X_1, \tau_1)$ and $\mathfrak{X}_2 = (X_2, \tau_2)$

$$\mathfrak{X}_1 \times \mathfrak{X}_2 = (X_1 \times X_2, \tau_1^*, \tau_2^*), \text{ where } \tau_1^* \text{ has base } \{U_1 \times x_2 \,|\, U_1 \in \tau_1 \;\&\; x_2 \in X_2\}$$
$$\tau_2^* \text{ has base } \{x_1 \times U_2 \,|\, x_1 \in X_1 \;\&\; U_2 \in \tau_2\}$$

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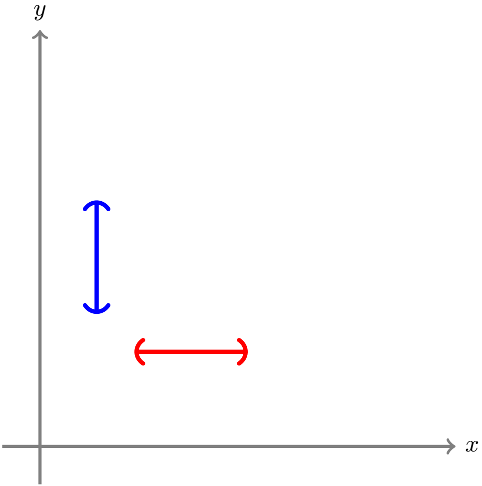

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For two logics L_1 and L_2

$$\mathsf{L}_1 \times_t \mathsf{L}_2 = Log(\{\mathfrak{X}_1 \times \mathfrak{X}_2 \,|\, \mathfrak{X}_1 \models \mathsf{L}_1 \;\&\; \mathfrak{X}_2 \models \mathsf{L}_2\})$$
$$\mathsf{S4} \times_t \mathsf{S4} = Log(\mathbb{Q} \times \mathbb{Q}) = \mathsf{S4} * \mathsf{S4} \quad \text{(van Benthem et al, 2005)}$$
$$Log(\mathbb{R} \times \mathbb{R}) \neq \mathsf{S4} * \mathsf{S4} \quad \text{(Kremer, 2010?)}$$
$$Log(Cantor \times Cantor) \neq \mathsf{S4} * \mathsf{S4}$$

d-logic of product of topological spaces was considered by L. Uridia (2011).

$$Log_d(\mathbb{Q} \times \mathbb{Q}) = \mathsf{D4} * \mathsf{D4}$$

Generalization to neighborhood frames was done by K. Sano (2011).

Known results

Theorem (2012)

Let L_1 *and* L_2 *be from the set* $\{\mathsf{D}, \mathsf{T}, \mathsf{D4}, \mathsf{S4}\}$ *then*

$$\mathsf{L}_1 \times_n \mathsf{L}_2 = \mathsf{L}_1 * \mathsf{L}_2.$$

Not straightforward but still a

Corollary

In derivational semantics

1. $\mathsf{D4} \times_d \mathsf{D4} = \mathsf{D4} * \mathsf{D4}$.
2. *[Uridia'2011]* $Log_d(\mathbb{Q} \times \mathbb{Q}) = \mathsf{D4} * \mathsf{D4}$

Note that all these logics include seriality: $\neg\Box\bot$.

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Without seriality

It is not the case for logic K!

Lemma

For any two n-frames $\mathfrak{X}_1$ and $\mathfrak{X}_2$

$$\mathfrak{X}_1 \times \mathfrak{X}_2 \models \Box_1\bot \to \Box_2\Box_1\bot.$$

And even more, for any closed $\Box_1$-free formula ϕ and any closed $\Box_2$-free formula ψ

$$\mathfrak{X}_1 \times \mathfrak{X}_2 \models \phi \to \Box_1\phi, \qquad \mathfrak{X}_1 \times \mathfrak{X}_2 \models \psi \to \Box_2\psi.$$

Proof.

$$\begin{aligned} \mathfrak{X}_1 \times \mathfrak{X}_2, (x, y) \models \Box_1\bot \iff \varnothing \in \tau_1'(x, y) \iff \\ \varnothing \in \tau_1(x) \iff \forall y' \in X_2\ (\varnothing \in \tau_1'(x, y')) \iff \\ \forall y' \in X_2\ (\mathfrak{X}_1 \times \mathfrak{X}_2, (x, y') \models \Box_1\bot) \implies \mathfrak{X}_1 \times \mathfrak{X}_2, (x, y) \models \Box_2\Box_1\bot. \end{aligned}$$

Hence, $\mathfrak{X}_1 \times \mathfrak{X}_2 \models \Box_1\bot \to \Box_2\Box_1\bot$. □

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Proof.

Since ψ does not contain neither $\Box_2$, nor variables, its value does not depend on the second coordinate. Let $F = \mathfrak{X}_1 \times \mathfrak{X}_2$. So $F, (x, y) \models \psi$, then $\forall y' (F, (x, y') \models \psi)$, hence, $F, (x, y) \models \Box_2 \psi$. $\Box$

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Definition

For two unimodal logics L_1 and L_2, we define weak commutator

$$\langle L_1, L_2 \rangle = L_1 * L_2 + \Delta, \text{ where}$$

$$\Delta = \{\phi \to \Box_2\phi \mid \phi \text{ is closed and } \Box_2\text{-free}\} \cup \{\psi \to \Box_1\psi \mid \psi \text{ is closed and } \Box_1\text{-free}\}.$$

Lemma

For any two normal modal logics L_1 and L_2 $\langle L_1, L_2 \rangle \subseteq L_1 \times_n L_2$.

Note that if $\Diamond\top \in L_1 \cap L_2$ then $L_1 * L_2 \models \Delta$.

Completeness results

Theorem (2014)

$\mathsf{K} \times_n \mathsf{K} = \langle \mathsf{K}, \mathsf{K} \rangle$.

Theorem

If logics L_1 *and* L_2 *are axiomatizable by closed formulas and by axioms like* $\Box p \to \Box^k p$ *then* $\mathsf{L}_1 \times_n \mathsf{L}_2 = \langle \mathsf{L}_1, \mathsf{L}_2 \rangle$.

Corollary

$\mathsf{K4} \times_d \mathsf{K4} = \langle \mathsf{K4}, \mathsf{K4} \rangle$.

Logic S5

We put

$$\begin{aligned} \Delta_1 &= \{\phi \to \Box_2\phi \,|\, \phi \text{ is closed and } \Box_2\text{-free}\}\,, \\ com_{12} &= \Box_1\Box_2 p \to \Box_2\Box_1 p, \\ com_{21} &= \Box_2\Box_1 p \to \Box_1\Box_2 p, \\ chr &= \Diamond_1\Box_2 p \to \Box_2\Diamond_1 p. \end{aligned}$$

Theorem

If logic L *is axiomatizable by closed formulas and by axioms like* $\Box p \to \Box^k p$ *then* $\mathsf{L} \times_n \mathsf{S5} = \mathsf{L} * \mathsf{S5} + \Delta_1 + com_{12} + chr$.

For $\mathsf{L} = \mathsf{S4}$ was proved by Kremer in 2011.

How to prove

PLAN

We have two logics L_1 and L_2. Let Γ_i are all axioms from L_i of form $\Box p \to \Box^k p$.

$$\text{Canonicity of the logic } \langle \mathsf{L}_1, \mathsf{L}_2 \rangle.$$
$$\Downarrow$$
$$\text{Construct } F_1 \models \mathsf{L}_1 \text{ and } F_2 \models \mathsf{L}_2, \text{ and } \langle F_1, F_2 \rangle \twoheadrightarrow \mathcal{F}_{\langle \mathsf{L}_1, \mathsf{L}_2 \rangle}, \text{ and } \langle F_1, F_2 \rangle \models \Delta.$$
$$\Downarrow$$
$$\text{Construct } \mathcal{N}_\omega{}^{\Gamma_1}(F_1) \times \mathcal{N}_\omega{}^{\Gamma_2}(F_2) \twoheadrightarrow \mathcal{N}\left(\langle F_1, F_2 \rangle^{\Gamma_1 \cup \Gamma_2}\right).$$
$$\Downarrow$$
$$\text{Check that } \mathcal{N}_\omega{}^{\Gamma_1}(F_1) \models \mathsf{L}_1 \text{ and } \mathcal{N}_\omega{}^{\Gamma_2}(F_2) \models \mathsf{L}_2$$

Here $\cdot^\Gamma$ is a special operation which makes sure that if $\Box p \to \Box^k p \in \Gamma$ then this formula is valid.

How to prove for S5

PLAN

We have two loogic L and S5

$$\text{Canonicity of the logic } \langle \mathsf{L}, \mathsf{S5}].$$
$$\Downarrow$$
$$\text{Construct } F_1 \models \mathsf{L} \text{ and } F_2 \models \mathsf{S5}, \text{ and } \langle F_1, F_2] \twoheadrightarrow \mathcal{F}_{\langle \mathsf{L}, \mathsf{S5}]}, \text{ and}$$
$$\langle F_1, F_2] \models \Delta_1, com_1, chr.$$
$$\Downarrow$$
$$\text{Construct } \mathcal{N}_\omega{}^\Gamma(F_1) \times \mathcal{N}_\omega{}^{\mathsf{S5}}(F_2) \twoheadrightarrow \mathcal{N}\left(\langle F_1, F_2]^\Gamma\right).$$

Take rooted frames $F_1 = (W_1, R_1)$ and $F_2 = (W_2, R_2)$ such that $W_1 \cap W_2 = \varnothing$ then

$$F_1 \between F_2 = \{x_1 x_2 \dots x_n \,|\, x_i \in W_1 \cup W_2 \text{ and projecton on } W_i \text{ is a path}\}$$

We define a Semi-Thue system

$$C_{12} = \{ab \mapsto ba \,|\, a \in W_1,\ b \in W_2\}$$

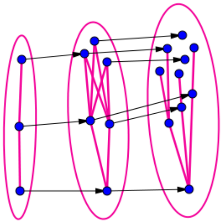
We also define a Kripke frame

$$\langle F_1, F_2] = (F_1 \between F_2, R_1^{<}, R_2^{\triangleleft})$$
$$\vec{a} R_1^{<} \vec{b} \iff \exists u \in W_1 (\vec{b} = \vec{a}u)$$
$$\vec{a} R_2^{<} \vec{b} \iff \exists v \in W_2 (\vec{b} = \vec{a}v)$$
$$\vec{a} R_2^{\triangleleft} \vec{b} \iff \exists \vec{b'} \,(\vec{a} R_2^{<} \vec{b'} \;\&\; \vec{b'} \underset{C_{12}}{\Longrightarrow} \vec{b})$$

Lemma

For F_1 and F_2 defined above

$$\langle F_1, F_2] \models com_{12},\ chr,\ \Delta_1.$$



THANK YOU!