Modal Logics of Infinite Depth

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§1. General Objective

My aim is to study lattices of extensions of modal logics. The generic case is Ext K. The structure of this lattice is approached here by awarding to every logic (if possible) an invariant, the depth. This depth may be infinite.

§2. Codimension and Depth

Let κ be an ordinal. A *downgoin chain* of type κ is a sequence $\langle L_\lambda : \lambda < \kappa + 1\rangle$ such that for every $\mu < \kappa + 1$ either $\mu = \mu' + 1$ and L_μ is a lower cover of $L_{\mu'}$ or μ is a limit ordinal and

$$L_\mu = \bigcap_{\mu' < \mu} L_{\mu'}.$$

L has finite depth if there is a finite downgoing chain ending in L. In a modular lattice, if L is of finite depth, any two downgoing chains have the same type. If the type is $n + 1$, the logic has depth n.

§3. Infinite Depth

Definition 1 *L has a depth if there is a downgoing chain ending in L. If L has a depth, the depth of L is the supremum of all κ such that a downgoing chain of type $\kappa + 1$ ending at L exists.*

L as a set if countable. If $M \prec L$ then $M \subsetneq L$. It follows that in modal logic, any downgoing chain must be countable. The supremum of a set of countable ordinals can be ω_1 at most. (ω_1 is the first oucountable ordinal.)

Lemma 1 *If L has a depth, the depth is less or equal to ω_1.*

§4. Splittings

- L is called $\bigsqcap$-*irreducible* if $\bigsqcap_{i \in I} M_i = L$ implies $M_i = L$ for some $i \in I$.
- L is called $\bigsqcap$-*prime* if $\bigsqcap_{i \in I} M_i \leq L$ implies $M_i \leq L$ for some $i \in I$.

(Analogously, $\bigsqcup$-*irreducible* and $\bigsqcup$-*prime* are defined.)

Lemma 2 *Let L be a modal logic. If $L' \geq L$ is $\bigsqcap$-prime in* Ext L *there exists a unique $\bigsqcup$-prime element L'' such that* Ext $L = \downarrow L' + \uparrow L''$. *$L'$ is called a splitting logic of* Ext L.

§5. Irreducibles

Let $\langle \mathrm{Irr}_L, \leq \rangle$ the poset of the $\sqcap$-irreducible logics in Ext L (where $\leq := \supseteq !$). Every logic L' above L determines a unique downward closed set $i(L')$ in Irr_L.

> Idea. If $L'' \prec L'$ then $i(L'') = i(L') \cup \{M\}$ for some M. Thus a downgoing chain of logics to L translate into a well-order extending $\leq$.

Since not all irreducibles are prime, this strategy needs to be exercised with care.

§6. WPOs

(The following draws on work by de Jongh and Parikh.)

A WPO is a partial order if (i) there are no infinite descending chains, and (ii) there are no infinite antichains.

Theorem 1 (de Jongh and Parikh) *If $\langle P, \leq \rangle$ is a WPO then there is a maximal well-order extending $\leq$. Denote it by $o(\langle P, \leq \rangle)$.*

Theorem 2 *If $\langle \mathrm{Irr}_L, \leq \rangle$ is a WPO, the depth of L is $\leq o(\langle P, \leq \rangle)$. If every $\sqcap$-irreducible logic is also $\sqcap$-prime, equality holds.*

§7. Logics without depth

If (i) is not satisfied (= there exists infinite descending chains in $\langle \mathrm{Irr}_L, \leq \rangle$), we cannot assign depth.

Theorem 3 *There is a logic L such that* $\mathrm{Ext}\ L \cong \omega + 2 + \omega^{op}$. $\langle \mathrm{Irr}_L, \leq \rangle$ *has infinite descending chains. L has no depth.*

§8. Infinite Antichains

If (ii) is not satisfied:

Theorem 4 *Assume L is the intersection of tabular logics. If* Ext L *contains an infinite antichain of splitting logics, L has depth ω_1.*

Corollary 1 $\mathsf{K}, \mathsf{K4}, \mathsf{S4}, \mathsf{G}, \mathsf{Grz}$ *have depth ω_1.*

§9. Logics with countable depth

Theorem 5 *Pretabular logics have depth* $\leq \omega$.

Theorem 6 *The logics* $\mathsf{K.Alt}_1$ *and* $\mathsf{K45}$ *have depth* $\omega + \omega$.

(Ext $\mathsf{K45}$ is continuous, but Ext $\mathsf{K.Alt}_1$ is not.)

Theorem 7 *The depth of* $\mathsf{S4.3}$ *is* $\omega^{\omega^\omega+1}$.

§10. Open Problems

Consider the sequence $1, \omega, \omega^\omega, \omega^{\omega^\omega}, \cdots$. Its limit is called ϵ_0.

Conjecture. If a logic has countable depth, this depth is $\leq \epsilon_0$.

Further, if there is a logic L of depth λ there also is a logic of depth $\mu < \lambda$, provided $\langle \mathrm{Irr}_L, \leq \rangle$ is a WPO. We have exhibited logics of depth $\leq \omega^{\omega^\omega+1}$. What about higher ordinals?

Problem. Construct logics of depth λ for countable λ (or show they cannot exist).

§11. Thank you!