Kripke Completeness of Strictly Positive Implications in Meet-Semilattices with Operators

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Strictly positive formulas and implications

Strictly positive formulas (SPF) are defined as

$$\phi ::= p_i \mid \bot \mid \top \mid \Diamond\phi \mid \phi \wedge \phi,$$

where p_i are propositional variables.

Strictly positive implications (SPI) are of the form

$$\phi_1 \to \phi_2,$$

where ϕ_1 and ϕ_2 are strictly positive.

Research problem

Given a normal modal logic $\mathcal{L}$, construct a calculus for its SPI-fragment (i.e., the set of all $\mathcal{SP}$-implications from $\mathcal{L}$).

Can we reuse the axioms of $\mathcal{L}$, if they already are SPIs?

Related Research

Description logic $\mathcal{EL}$ and medical ontologies

SNOMED CT contains $\geq$ 300000 implications like

$$\text{KidneyDisease} \equiv \text{Disorder} \sqcap \exists \text{findingSite}.\text{KidneyStructure};$$

A huge number of both theoretical and practical paper about numeruous reasoning tasks with such axioms.

Strictly Positive fragments of provability logics

L. Beklemishev, E. Dashkov studied SPI-fragments of the logic *GLP*;
Svyatlovsky studied SPI-fragment of **K4.3**.

Reseach in meet-semilattice algebras

M. Jackson considered semilattices with closure related to the extensions of **S5**.

Distributive modal logic

(Goldblatt, 1989) and (M. Gherke, H. Nagahashi, Y. Venema, 2005) showed that a version of Sahlqvist completeness theorem holds if we remove negation from the basic modal language. What if we in addition remove disjunction?

Kripke completeness of SPIs

Two semantics for SPIs

Strictly positive formulas and implications may be interpreted:

- on Kripke frames; $\mathcal{E} \models_{Kr} \mathbf{e}$ is the consequence relation on all Kripke frames;
- on meet-semilattices with monotone operators (SLOs) (or 'general' frames); $\mathcal{E} \models_{SLO} \mathbf{e}$ is the consequence relation on all such structures.

Main definition

An *SPI*-theory $\mathcal{E}$ is complete, if for all SP implications $\mathbf{e}$ we have

$$\mathcal{E} \models_{Kr} \mathbf{e} \iff \mathcal{E} \models_{SLO} \mathbf{e}.$$

(in this case the *SPI*-fragment of $\mathbf{K} + \mathcal{E}$ is axiomatised by $\mathcal{E}$ with standard SLO axioms)

Examples

- $\{\}$ is complete (folklore)
- $\{p \to \Diamond p, \Diamond\Diamond p \to \Diamond p\}$ is complete (Jackson, 2004)
- any set of implications of the form $\Diamond_1 \dots \Diamond_n p \to \Diamond_0 p$ is complete (Sofronie-Stokkermans, 2008)

How does incompleteness occur ?

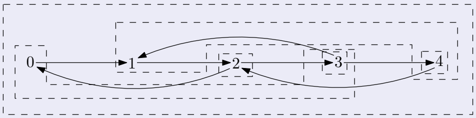
'Reversed transitivity'

Implication $\mathbf{e} = p \wedge \Diamond\Diamond q \to \Diamond\Diamond(q \wedge \Diamond p)$ with FO equivalent

$\forall x \forall y \forall z (R(x,y) \wedge R(y,z) \to R(z,x))$

is incomplete, since $\mathbf{e} \models_{Kr} \Diamond\Diamond\Diamond\Diamond p \to \Diamond p$

but $\mathbf{e} \not\models_{SLO} \Diamond\Diamond\Diamond\Diamond p \to \Diamond p$



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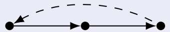

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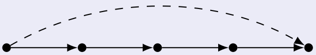

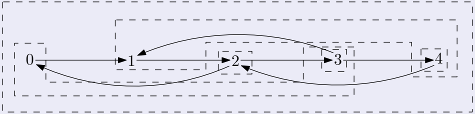

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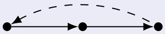

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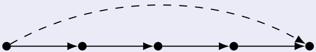
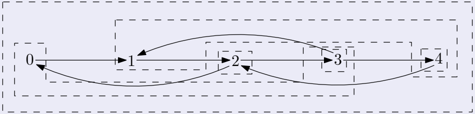




But

Implication $p \wedge \Diamond q \rightarrow \Diamond(q \wedge \Diamond p)$ expressing symmetry  is complete.

What remains of the Sahlqvist theorem ?

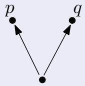
Theorem

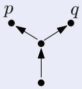
Any $\mathcal{EL}$-theory $\mathcal{E}$ consisting of equations $\mathbf{e} = (\sigma \to \tau)$ such that

- *every variable in σ occurs in it only once,*
- *τ corresponds to the tree $\mathfrak{T}_\tau = (W_\tau, R_\tau, V_\tau)$ with*
  - *$|W_\tau| \geq 2$ and all points in some $V_\tau(p)$ are leaves of $\mathfrak{T}_\tau$,*
  - *$V_\tau(p) \cap V_\tau(q) = \emptyset$ whenever $p \neq q$*

is complete.

Example





$\Diamond p \wedge \Diamond q \quad \to \quad \Diamond(\Diamond p \wedge \Diamond q)$

Proof

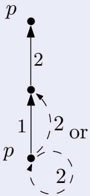
Similar to the Jonsson-Tarski construction: we embed SLOs satisfying $\mathcal{E}$ into Kripke frames with needed properties using filters (or even upward-closed sets) instead of ultrafilters.

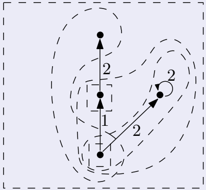
Applied to:

reflexivity, transitivity, (generalised) density, standard rooted Horn formulas

Disjunction on the right-hand side of FO-equivalents is another reason of incompleteness:

The implication $\mathbf{e} = (p \wedge \Diamond_1 p \to \Diamond_2 p)$ with FO-equivalent $\forall x, y\,(R_1(x,y) \to R_2(x,x) \vee R_2(x,y))$ is not complete since $\mathbf{e} \models_{Kr} p \wedge \Diamond_1\Diamond_2 p \to \Diamond_2\Diamond_2 p$, but $\mathbf{e} \not\models_{SLO} p \wedge \Diamond_1\Diamond_2 p \to \Diamond_2\Diamond_2 p$:





However,

SPI-axiomatisation $\mathcal{E}$ of **S4.3**: $p \to \Diamond p \qquad \Diamond\Diamond p \to \Diamond p$

$$\Diamond(p \wedge q) \wedge \Diamond(p \wedge r) \to \Diamond(p \wedge \Diamond q \wedge \Diamond r)$$

- $\mathcal{E}$ is Kripke complete (can be proved via nice explicit description of SPI-consequences of $\mathcal{E}$).
- Not every $\mathcal{E}$-SLO is embeddable to the complex algebra of an **S4.3**-frame.

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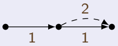
So what SPI-theories are complete and what are not ?

$\Diamond_1(p \wedge \Diamond_1 q) \rightarrow \Diamond_1(p \wedge \Diamond_2 q)$ with profile 

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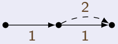
$\Diamond_1(p \wedge \Diamond_1 q) \to \Diamond_1(p \wedge \Diamond_2 q)$ with profile 

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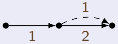
while $\Diamond_1(p \wedge \Diamond_2 q) \to \Diamond_1(p \wedge \Diamond_1 q)$ with profile 

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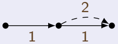
$\Diamond_1(p \wedge \Diamond_1 q) \rightarrow \Diamond_1(p \wedge \Diamond_2 q)$ with profile 

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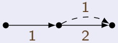
while $\Diamond_1(p \wedge \Diamond_2 q) \rightarrow \Diamond_1(p \wedge \Diamond_1 q)$ with profile 

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$p \rightarrow \Diamond\Diamond(p \wedge \Diamond p)$ expressing reflexivity is incomplete;

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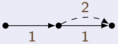
$\Diamond_1(p \wedge \Diamond_1 q) \to \Diamond_1(p \wedge \Diamond_2 q)$ with profile 

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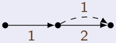
while $\Diamond_1(p \wedge \Diamond_2 q) \to \Diamond_1(p \wedge \Diamond_1 q)$ with profile 

is not;

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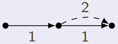
$\Diamond_1(p \wedge \Diamond_1 q) \to \Diamond_1(p \wedge \Diamond_2 q)$ with profile 

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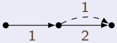
while $\Diamond_1(p \wedge \Diamond_2 q) \to \Diamond_1(p \wedge \Diamond_1 q)$ with profile 

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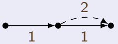
$\Diamond_1(p \wedge \Diamond_1 q) \to \Diamond_1(p \wedge \Diamond_2 q)$ with profile 

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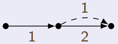
while $\Diamond_1(p \wedge \Diamond_2 q) \to \Diamond_1(p \wedge \Diamond_1 q)$ with profile 

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all SPI-theories which axiomatise the extensions of **S5** except one are complete;

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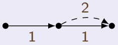
$\Diamond_1(p \wedge \Diamond_1 q) \to \Diamond_1(p \wedge \Diamond_2 q)$ with profile 

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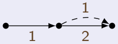
while $\Diamond_1(p \wedge \Diamond_2 q) \to \Diamond_1(p \wedge \Diamond_1 q)$ with profile 

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Theorem

It is undecidable whether an SPI-theory is Kripke complete.