Decidable first-order modal logics with counting quantifiers

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Background

First-order modal logics with counting quantifiers

- **Formulas:** $\varphi ::= P_i(x_1, \ldots, x_n) \mid \neg\varphi \mid (\varphi_1 \wedge \varphi_2) \mid \Diamond\varphi \mid (\exists_{\leq c} x\, \varphi)$

- **First-order Kripke models:** $\mathfrak{M} = (\mathfrak{F}, D, I)$
  - Kripke frame $\mathfrak{F} = (W, R)$
  - Non-empty domain $D = \{$ domains objects $\}$
  - Interpretation function

$$I(w) = \left\langle D, P_0^{I(w)}, P_1^{I(w)}, \ldots \right\rangle$$

- **Satisfiability:**

$$\mathfrak{M}, w \models^{\mathfrak{a}} (\exists_{\leq c} x\, \varphi) \quad \text{iff} \quad \left|\{b \in D \,:\, \mathfrak{M}, w \models^{\mathfrak{a}(x/b)} \varphi\}\right| \leq c$$

(where $\mathfrak{a}(x/b)(x) = b$ and $\mathfrak{a}(x/b)(y) = \mathfrak{a}(y)$, for $y \neq x$.)

First-order modal logics with counting quantifiers

- **Logics with counting quantifiers**

$$\mathbf{Q^{\#}Log}(\mathcal{C}) = \{ \text{ formulas valid in all frames } \mathfrak{F} \in \mathcal{C} \ \}$$

  - **Some examples:**

$$\begin{aligned}
\mathbf{Q^{\#}K} &= \mathbf{Q^{\#}Log}\{ \text{ all frames } \} \\
\mathbf{Q^{\#}KT} &= \mathbf{Q^{\#}Log}\{ \text{ all reflexive frames } \} \\
\mathbf{Q^{\#}KB} &= \mathbf{Q^{\#}Log}\{ \text{ all symmetric frames } \} \\
\mathbf{Q^{\#}S5} &= \mathbf{Q^{\#}Log}\{ \text{ all equivalence relations } \} \\
\mathbf{Q^{\#}Alt} &= \mathbf{Q^{\#}Log}\{ \text{ all partial functions } \}
\end{aligned}$$

First-order modal logics with counting quantifiers

- **ℓ-variable fragment:**

$$\mathcal{Q}^{\#}\mathcal{ML}^{\ell} = \{\varphi \in \mathcal{Q}^{\#}\mathcal{ML} \ : \ \varphi \text{ contains only } x_1, \ldots, x_\ell \}$$

- **k-bounded fragment**

$$\mathcal{Q}^{\#}\mathcal{ML}_k = \{\varphi \in \mathcal{Q}^{\#}\mathcal{ML} \ : \ \varphi \text{ contains only } (\exists_{\leq c} x_i) \text{ for } c \leq k\}$$

  - **zero-bounded = regular FOL** $\mathcal{Q}^{\#}\mathcal{ML}_0$

First-order modal logics with counting quantifiers

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  - zero-bounded = regular FOL $\mathcal{Q}^{\#}\mathcal{ML}_{0}$

- k-bounded, ℓ-variable fragment

$$\mathcal{Q}^{\#}\mathcal{ML}^{\ell}_{k} = \mathcal{Q}^{\#}\mathcal{ML}^{\ell} \cap \mathcal{Q}^{\#}\mathcal{ML}_{k}$$

Finite model property

- **Finite model property (fmp):**

$$\varphi \text{ is } \mathbf{Q}^{\#}\boldsymbol{L}\text{-satisfiable} \quad \Longrightarrow \quad (\mathfrak{F}, D, I), w \models^{\mathfrak{a}} \varphi, \text{ for } \mathfrak{F} \in \textbf{Fr } \boldsymbol{L}$$

where:

- **Finite (fmp):** $|\mathfrak{F}|$ and $|D|$ are both **finite**,
- **Poly-size fmp:** $|\mathfrak{F}|$ and $|D|$ are **polynomial** in the size of φ,
- **Exponential fmp:** $|\mathfrak{F}|$ and $|D|$ are **exponential** in the size of φ,

Two-dimensional modal logics

- **Bimodal formulas:** $\varphi \ ::= \ p \mid \neg\varphi \mid (\varphi_1 \wedge \varphi_2) \mid \Diamond_h \varphi \mid \Diamond_v \varphi$

- **Kripke models:** $\mathfrak{M} = (\mathfrak{F}, \mathfrak{V})$

  - Kripke 2-frame $\mathfrak{F} = (W, R_h, R_v)$

  - Propositional valuation $\mathfrak{V}(p) = \{$ domains objects $\}$

- **Logic of $\mathcal{C}$:**

$$\mathbf{Log}(\mathcal{C}) = \{ \text{ formulas valid in all frames } \mathfrak{F} \in \mathcal{C} \ \}$$

Two-dimensional modal logics

Shehtman 1978, Segerberg 1973

The **product frame** of $\mathfrak{F}_h = (W_h, R_h)$ and $\mathfrak{F}_v = (W_v, R_v)$ is the 2-frame

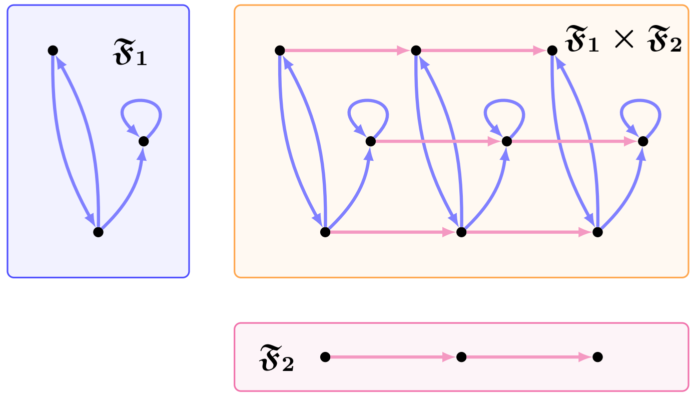
$$\mathfrak{F}_h \times \mathfrak{F}_v = \big(W_h \times W_v, \bar{R}_h, \bar{R}_v\big)$$

where

$$(x, y)\bar{R}_h(x', y') \iff x R_h x' \text{ and } y = y'$$

and

$$(x, y)\bar{R}_v(x', y') \iff x = x' \text{ and } y R_v y'$$



The **product** of two Kripke complete unimodal logics L_1, L_2 is the bimodal logic

$$L_1 \times L_2 = \mathbf{Log}\{\mathfrak{F}_h \times \mathfrak{F}_v \ : \ \mathfrak{F}_h \models L_1 \text{ and } \mathfrak{F}_v \models L_2\}$$

Connection with first-order modal logics

- **Zero-bounded (counting free) fragment:** $\mathbf{Q}^{\#}L \cap \mathcal{Q}^{\#}\mathcal{ML}_0^1$
  - **Classical case:** $\mathbf{S5} = \mathbf{Log}\{$ all equivalence relations $\}$

$p_i \rightsquigarrow P_i(x)$ $\qquad$ $\Diamond\psi \rightsquigarrow (\exists x\ \psi)$

$\mathbf{S5} \leftrightsquigarrow$ classical FOL

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  - **Modal case:**

    $\Diamond_h\psi \rightsquigarrow \Diamond\psi$ $\qquad$ $\Diamond_v\psi \rightsquigarrow (\exists x\ \psi)$

    $L \times \mathbf{S5} \leftrightsquigarrow \mathbf{Q}^{\#}L \cap \mathcal{Q}^{\#}\mathcal{ML}_0^1$

Connection with first-order modal logics

- **One-bounded fragment:** $\mathbf{Q}^{\#}\boldsymbol{L} \cap \boldsymbol{\mathcal{Q}}^{\#}\boldsymbol{\mathcal{ML}}_1^1$
  - **Classical case:** **Diff** $=$ **Log**$\{$ all difference frames $(\boldsymbol{W}, \neq)$ $\}$

$$\Diamond\psi \quad \rightsquigarrow \quad (\exists^{\neq}\boldsymbol{x}\ \psi) \ := \ (\neg\psi \land \exists_{>\mathbf{0}}\boldsymbol{x}\ \psi) \lor \exists_{>\mathbf{1}}\psi$$

Diff $\quad\leftrightsquigarrow\quad$ classical FOL $+\ \exists_{\leq\mathbf{0}},\ \exists_{\leq\mathbf{1}}$

Connection with first-order modal logics

- **One-bounded fragment:** $\mathbf{Q}^{\#}L \cap \mathcal{Q}^{\#}\mathcal{ML}_1^1$

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    Diff $\leftrightsquigarrow$ classical FOL $+\ \exists_{\leq 0},\ \exists_{\leq 1}$

  - **Modal case:**

    $\Diamond_h\psi \quad \rightsquigarrow \quad \Diamond\psi$ $\qquad$ $\Diamond_v\psi \quad \rightsquigarrow \quad (\exists^{\neq} x\ \psi)$

    $$L \times \mathbf{Diff} \quad \leftrightsquigarrow \quad \mathbf{Q}^{\#}L \cap \mathcal{Q}^{\#}\mathcal{ML}_1^1$$

The story so far...

- *Kripke 1962*

  The **two-variable monadic** fragment is **undecidable**.

- *Marx 1999*

  $\boldsymbol{L} \times \mathbf{S5}$ is **coNExpTime-hard** $(\mathbf{K} \subseteq \boldsymbol{L} \subseteq \mathbf{S5})$,

  $\rightsquigarrow$ $\mathbf{Q}^{\#}\boldsymbol{L} \cap \mathcal{Q}^{\#}\mathcal{ML}_0^1$ is **coNExpTime-hard**

- *Wolter–Zakharyaschev 2001*

  The **monodic** fragment is **decidable**.

- *Pratt-Hartmann 2005*

  Two-variable FOL + counting is **coNExpTime-complete**,

  $\rightsquigarrow$ $\mathbf{Q}^{\#}\mathbf{S5} \cap \mathcal{Q}^{\#}\mathcal{ML}^1$ is **coNExpTime-complete**.

The story so far...

- *H–Kurucz 2012*

  'K + universal operator' $\times$ Diff is **undecidable**

  $\rightsquigarrow$ $\mathbf{Q}^{\#}\mathbf{K}^{\forall} \cap \mathcal{Q}^{\#}\mathcal{ML}_1^1$ is **undecidable**

  'K + transitive closure' $\times$ Diff is **non-r.e.**

  $\rightsquigarrow$ $\mathbf{Q}^{\#}\mathbf{K}^{*} \cap \mathcal{Q}^{\#}\mathcal{ML}_1^1$ is **non-r.e.**

- *H–Kurucz 2015*

  'linear' $\times$ Diff usually **undeciable** (or worse!)

  $\rightsquigarrow$ $\mathbf{Q}^{\#}\mathbf{K4.3} \cap \mathcal{Q}^{\#}\mathcal{ML}_1^1$ is **undecidable**

  $\mathbf{Q}^{\#}\mathbf{Log}(\mathbb{N}) \cap \mathcal{Q}^{\#}\mathcal{ML}_1^1$ is Π_1^1**-hard**

Main results

Main results I

Theorem The one-variable fragment of $\mathbf{Q}^{\#}\mathbf{K}$:

(i) has the **exponential** fmp,

(ii) is **coNExpTime-complete**.

Proof (overview):

Step 1) Define an appropriate notion of a **'quasimodel'**

Step 2) Establish the equivalance

$\exists$ Quasimodel iff $\mathbf{Q}^{\#}\mathbf{K}$-satisfiable

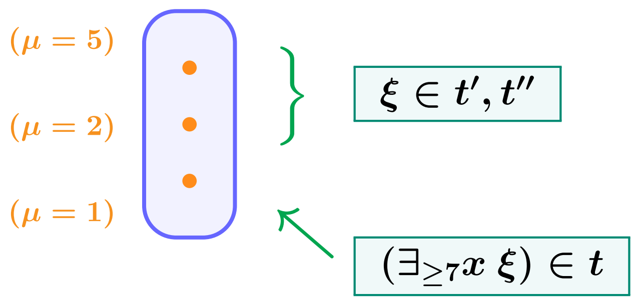
Step 3) Procedure to **'prune'** large quasimodels

Quasimodel size $= O(2^{||\varphi||})$

Quasistates and Quasimodels

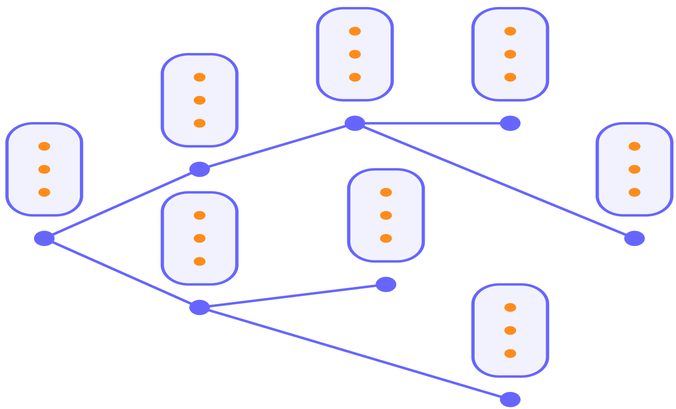
- **Quasistate:** (T, μ)
  - **Domain** $T = \{$ Boolean saturated types $\}$
  - **(Bounded) Multiplicity function** $\mu : T \to \{1, \ldots, C, C+1\}$
  - **Saturation criterion:**

$$(\exists_{\leq c} x\, \xi) \in t \quad \text{iff} \quad \textstyle\sum_{\xi \in t'} \mu(t') \leq c$$



Quasistates and Quasimodels

- **Quasimodel:** $\mathfrak{Q} = (W, \prec, \boldsymbol{q}, \mathfrak{R})$
  - Intransitive, irreflexive tree $(W, \prec)$
  - Quasistate assignment $\boldsymbol{q}(w) = (T_w, \mu_w)$
  - Set of (indexed) runs $r(w) \in T_w$ for all $w \in W$



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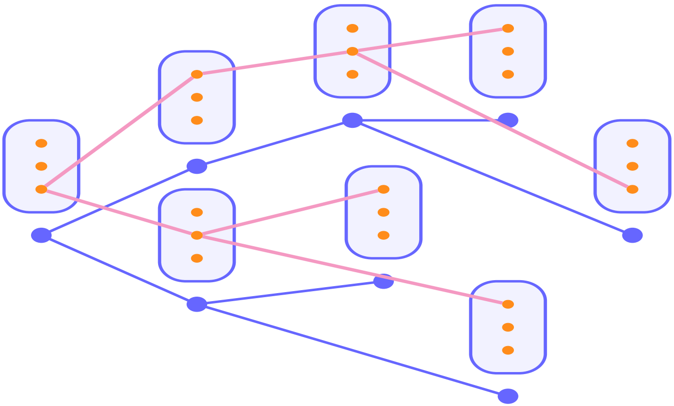

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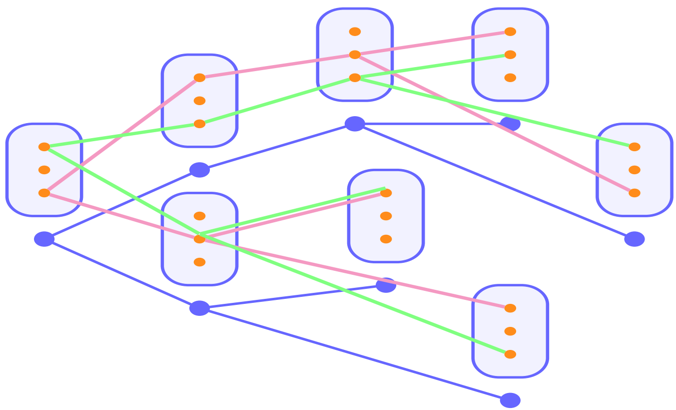
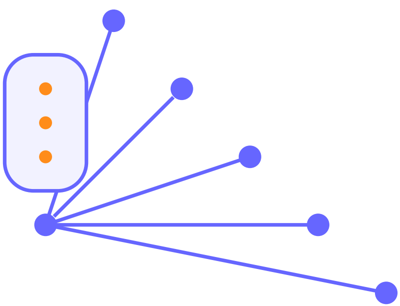


Pruning Procedure

Step 1a) Build a ‘small’ **subtree of witnesses**,



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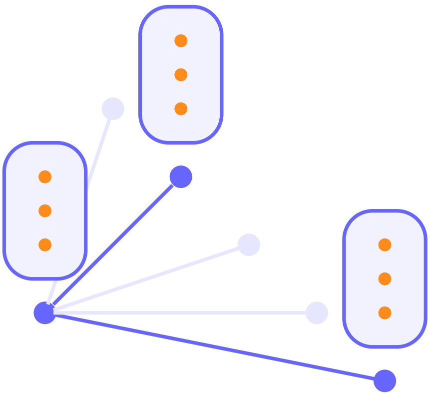

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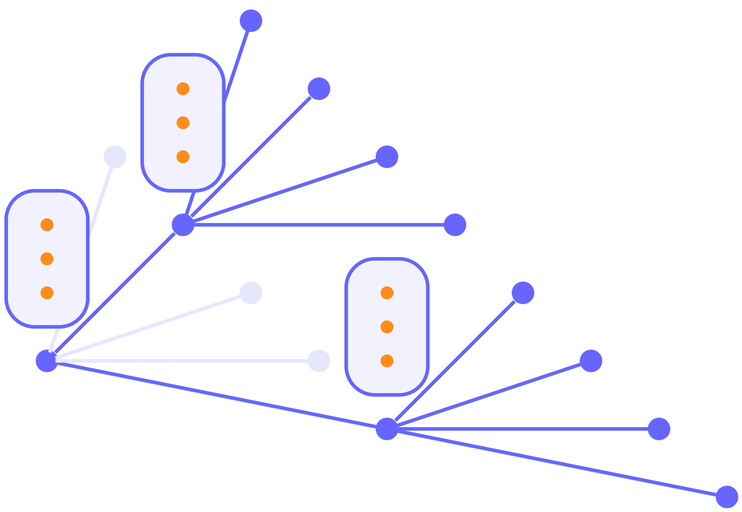

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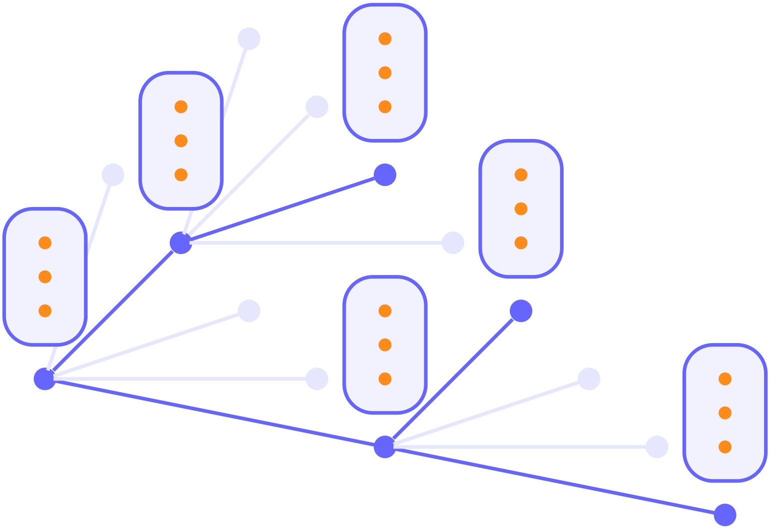
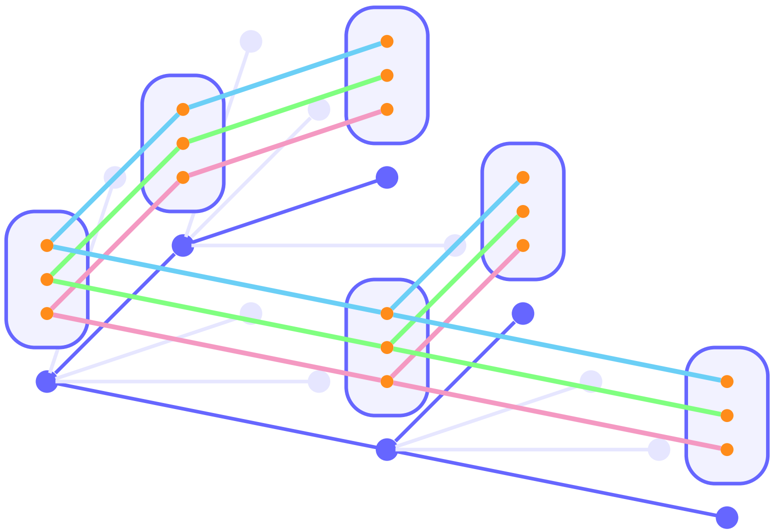


Pruning Procedure

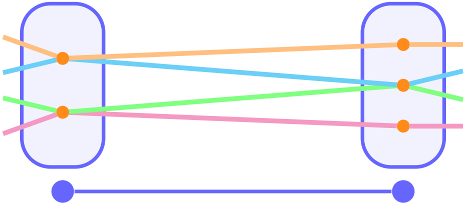
Step 1a) Build a 'small' **subtree of witnesses**,

Step 1b) **Saturate** with 'sufficiently' many runs,



Pruning Procedure

Step 2a) **Clone** each subtree 'sufficiently' many times,



Pruning Procedure

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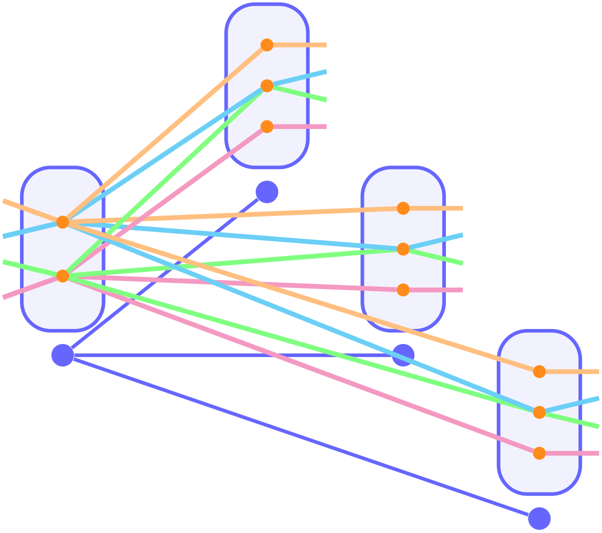
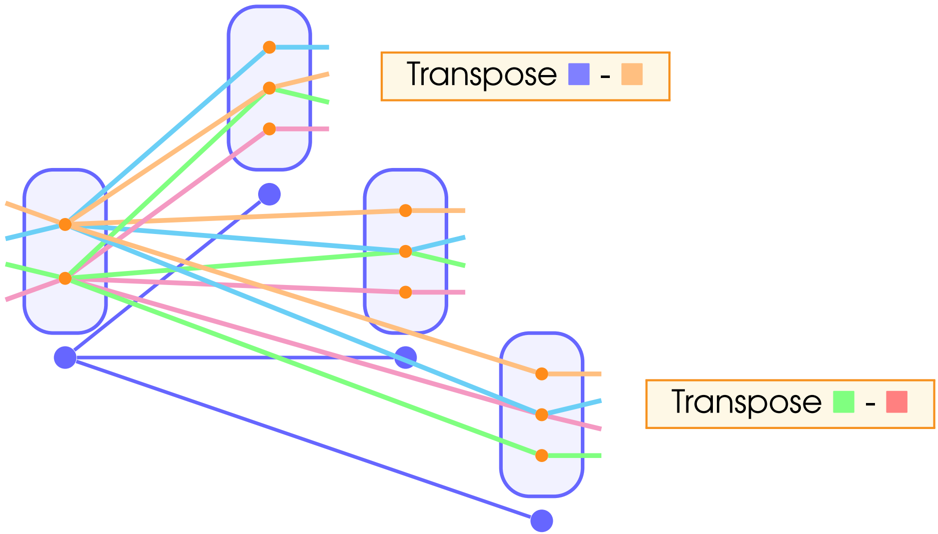


Pruning Procedure

Step 2a) **Clone** each subtree 'sufficiently' many times,

Step 2b) **Repair saturation** criterion by transposing runs in cloned states



Main results II

Theorem The one-variable fragment of $\mathbf{Q}^{\#}\boldsymbol{L}$:

(i) has the **exponential** fmp,

(ii) is **coNExpTime-complete**.

for $\boldsymbol{L} \in \{\mathbf{KT}, \mathbf{KB}, \mathbf{S5}\}$.

Proof (overview):

Define a **model-level reduction** for each $\boldsymbol{L} \in \{\mathbf{KT}, \mathbf{KB}, \mathbf{S5}\}$,

- **Translation** $(\cdot)^{\dagger} : \mathcal{Q}^{\#}\mathcal{ML}^{1} \to \mathcal{Q}^{\#}\mathcal{ML}^{1}$
- **Frame Transformation** $(\cdot)^{\star} : \text{Frames}(\mathbf{K}) \to \text{Frames}(\boldsymbol{L})$

$$\mathfrak{F} \not\models \varphi^{\dagger} \quad \text{iff} \quad \mathfrak{F}^{\star} \not\models \varphi$$

Main results III

Theorem The one-variable fragment of $\mathbf{Q}^{\#}\mathbf{Alt}$:

(i) has the **poly-size** fmp,

(ii) is **coNP-complete**.

Proof (overview):

Obsv 1) Every $\mathbf{Q}^{\#}\mathbf{Alt}$-satisfiable formula has a model of **depth** $\leq \mathbf{md}(\varphi)$.

Obsv 2) Modal depth can be **'flattened'** to yield $\mathbf{t}_\ell(\varphi) \in \mathcal{C}^1$

φ is satisfiable in model of depth ℓ iff $\mathbf{t}_\ell(\varphi)$ is FO-satisfiable

($\mathcal{C}^1$ = one-variable fragment of classical FOL with counting quantifiers)

Obsv 3) The one-variable fragment $\mathcal{C}^1$ is **NP-complete**.

Pratt-Hartmann 2005

Conclusion

Open problems

Q: Is the **monodic fragment** of $\mathbf{Q}^{\#}\mathbf{K}$ appropriately extended with counting quantifiers decidable?

- **Note:** No obvious application of *Wolter–Zakharyaschev 1998*

  $\because$ $\mathbf{Q}^{\#}\mathbf{K}^{*} \cap \mathcal{Q}^{\#}\mathcal{ML}^{1}$ is **non-r.e.**

  H–Kurucz 2012

Q: Is the one-variable fragment of $\mathbf{Q}^{\#}\mathbf{K4}$ ('transitive frames') decidable?

Q: The one-variable fragment of $\mathbf{Q}^{\#}\mathbf{K4.3}$ ('linear orders') is **undecidable**

H–Kurucz 2015

- Does this remain true over **expanding domains**?

Thank you for listening!

Some references

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