Interval Temporal Logics: a selective overview

Dedicated to the memory of Sasha Chagrov

Valentin Goranko
Department of Philosophy, Stockholm University

Advances in Modal Logic 2016
Budapest, August 30, 2016

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Outline

Introduction. Interval structures and relations

Interval temporal logics

Halpern-Shoham's logic HS and its fragments

Relative expressiveness of the fragments of HS

Undecidability

Decidability

Representation theorems and axiomatic systems

Interval-based temporal reasoning: origins and applications

Interval-based temporal reasoning: reasoning about time, where the primary concept is 'time interval', rather than 'time instant'.

Origins:

- Philosophy, in particular philosophy and ontology of time.
- Linguistics: analysis of progressive tenses, semantics of natural languages.
- Artificial intelligence: temporal knowledge representation, temporal planning, theory of events, etc.
- Computer science: specification and design of hardware systems, concurrent real-time processes, temporal databases, etc.

Preliminaries: Intervals and interval structures

$\mathbf{D} = \langle D, < \rangle$: partially ordered set.

An interval in $\mathbf{D}$: ordered pair $[a, b]$, where $a, b \in D$ and $a \leq b$.

If $a < b$ then $[a, b]$ is a strict interval; $[a, a]$ is a point interval.

$\mathbb{I}^+(\mathbf{D})$: the (non-strict) interval structure over $\mathbb{D}$, consisting of the set of all intervals over $\mathbf{D}$.

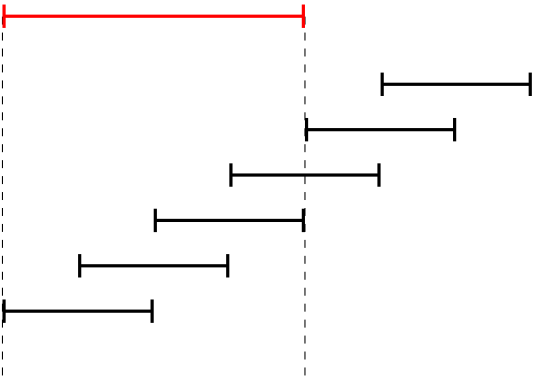
$\mathbb{I}^-(\mathbf{D})$ the strict interval structure over $\mathbb{D}$, consisting of the set of all strict intervals over $\mathbf{D}$.

We will use $\mathbb{I}(\mathbf{D})$ to denote either of these.

In this talk I will restrict attention to linear interval structures, i.e. interval structures over linear orders. Many of the results extend to partial orders with the linear intervals property.

Binary interval relations on linear orders

J. F. Allen: Maintaining knowledge about temporal intervals.
Communications of the ACM, volume 26(11), pages 832-843, 1983.

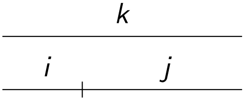


Later
After (right neighbour)
Overlaps (on the right)
Ends (the current interval)
During (the current interval)
Begins (the current interval)

6 relations + their inverses + equality = 13 Allen's relations.

Ternary relations between intervals

Splitting of an interval in two defines the ternary relation Chop:



i.e., $Cijk$ if i meets j, i begins k, and j ends k.

The relation Chop has 5 associated 'residual' relations, e.g.:

$$Dijk \text{ iff } Cikj,$$

$$Tijk \text{ iff } Ckij.$$

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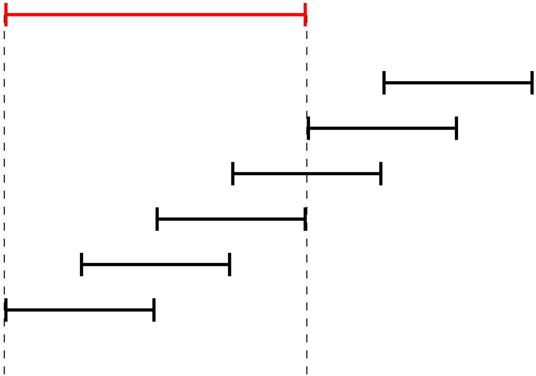
Halpern-Shoham's modal logic of interval relations HS

Every binary interval relation gives rise to a unary modal operator over relational interval structures. Thus, a multimodal logic arises:



Later	$\langle L \rangle$	$\langle \overline{L} \rangle$
After	$\langle A \rangle$	$\langle \overline{A} \rangle$
Overlaps	$\langle O \rangle$	$\langle \overline{O} \rangle$
Ends	$\langle E \rangle$	$\langle \overline{E} \rangle$
During	$\langle D \rangle$	$\langle \overline{D} \rangle$
Begins	$\langle B \rangle$	$\langle \overline{B} \rangle$

J. Halpern and Y. Shoham: A propositional modal logic of time intervals. *Journal of the ACM*, volume 38(4), pages 935-962, 1991.

Other important interval logics

▷ Moszkowski's (1983) Propositional Interval Temporal Logic (PITL)

Formulae: $\phi ::= p \mid \neg\phi \mid \phi \wedge \psi \mid \bigcirc \phi \mid \phi ; \psi$,
where $\bigcirc$ is Nexttime and ; is Chop.

Models of PITL: based on discrete linear orderings.

PITL-formulae are evaluated on discrete intervals:
finite sequences of *states* $\sigma = s_0, s_1, \ldots, s_n$, with $n \geq 0$.

▷ Zhou, Hoare, & Ravn's Duration calculus (DC): extension of the PITL framework with the notions of state and state duration.

▷ Venema's logic CDT involving binary modalities **C**, **D**, and **T**, associated with the relation Chop and its residuals.

Hereafter I will focus on Halpern-Shoham's logic HS and its fragments.
Denoted by listing the occurring modalities, e.g. D, BE, $O\overline{B}A\overline{A}$, etc.

Main topics on fragments of HS and current state of the art

- **Classification and relative expressiveness**.
  Current state: almost completed.
- **Decidability / undecidability** of the validities.
  Current state: Most (>90%) of HS' fragments are undecidable.
  But, there are several non-trivial decidability results.
  Decision methods: mainly tableau-based.
- **Representation theorems** for interval structures.
  Current state: A number of results, but many open cases, too.
  Not a systematic picture yet.
- **Axiomatic systems and completeness results**.
  Current state: Few results. Not much explored.
- **Model checking**.
  Current state: Still early stage. Some interesting developments.
- **Extensions**.
  Current state: mainly *metric* and *two-sorted* (point-interval).

Strict and non-strict models for propositional interval logics

$\mathcal{AP}$: a set of atomic propositions (over intervals).

Non-strict interval model:

$$\mathbf{M}^+ = \langle \mathbb{I}(\mathbb{D})^+, V \rangle,$$

where $V : \mathcal{AP} \mapsto 2^{\mathbb{I}(\mathbb{D})^+}$.

Strict interval model:

$$\mathbf{M}^- = \langle \mathbb{I}(\mathbb{D})^-, V \rangle,$$

where $V : \mathcal{AP} \mapsto 2^{\mathbb{I}(\mathbb{D})^-}$.

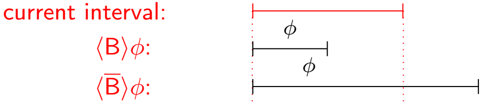
Thus, every $V(p)$ can be viewed as a binary relation on D.

Hereafter $\mathbf{M}$ will denote a strict or a non-strict interval model.

Formal semantics of HS

$\langle B \rangle$: $\mathbf{M}, [d_0, d_1] \Vdash \langle B \rangle \phi$ iff there exists d_2 such that $d_0 \leq d_2 < d_1$ and $\mathbf{M}, [d_0, d_2] \Vdash \phi$.

$\langle \overline{B} \rangle$: $\mathbf{M}, [d_0, d_1] \Vdash \langle \overline{B} \rangle \phi$ iff there exists d_2 such that $d_1 < d_2$ and $\mathbf{M}, [d_0, d_2] \Vdash \phi$.



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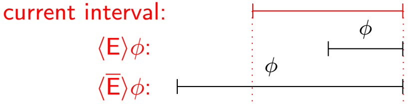
$\langle E \rangle$: $\mathbf{M}, [d_0, d_1] \Vdash \langle E \rangle \phi$ iff there exists d_2 such that $d_0 < d_2 \leq d_1$ and $\mathbf{M}, [d_2, d_1] \Vdash \phi$.

$\langle \overline{E} \rangle$: $\mathbf{M}, [d_0, d_1] \Vdash \langle \overline{E} \rangle \phi$ iff there exists d_2 such that $d_2 < d_0$ and $\mathbf{M}, [d_2, d_1] \Vdash \phi$.



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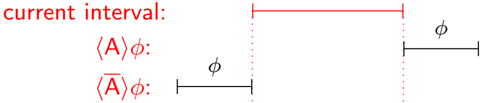
$\langle \overline{E} \rangle$: $\mathbf{M}, [d_0, d_1] \Vdash \langle \overline{E} \rangle \phi$ iff there exists d_2 such that $d_2 < d_0$ and $\mathbf{M}, [d_2, d_1] \Vdash \phi$.

$\langle A \rangle$: $\mathbf{M}, [d_0, d_1] \Vdash \langle A \rangle \phi$ iff there exists d_2 such that $d_1 < d_2$ and $\mathbf{M}, [d_1, d_2] \Vdash \phi$.

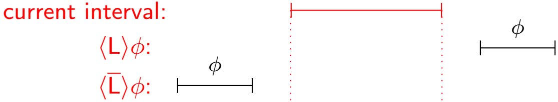
$\langle \overline{A} \rangle$: $\mathbf{M}, [d_0, d_1] \Vdash \langle \overline{A} \rangle \phi$ iff there exists d_2 such that $d_2 < d_0$ and $\mathbf{M}, [d_2, d_0] \Vdash \phi$.



Formal semantics of HS - contd'

$\langle L \rangle$: $\mathbf{M}, [d_0, d_1] \Vdash \langle L \rangle \phi$ iff there exists d_2, d_3 s.t. $d_1 < d_2 < d_3$ and $\mathbf{M}, [d_2, d_3] \Vdash \phi$.

$\langle \overline{L} \rangle$: $\mathbf{M}, [d_0, d_1] \Vdash \langle \overline{L} \rangle \phi$ iff there exists d_2, d_3 s.t. $d_2 < d_3 < d_0$ and $\mathbf{M}, [d_2, d_3] \Vdash \phi$.



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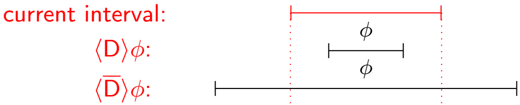
$\langle \overline{L} \rangle$: $\mathbf{M}, [d_0, d_1] \Vdash \langle \overline{L} \rangle \phi$ iff there exists d_2, d_3 s.t. $d_2 < d_3 < d_0$ and $\mathbf{M}, [d_2, d_3] \Vdash \phi$.

$\langle D \rangle$: $\mathbf{M}, [d_0, d_1] \Vdash \langle D \rangle \phi$ iff there exists d_2, d_3 s.t. $d_0 < d_2 < d_3 < d_1$ and $\mathbf{M}, [d_2, d_3] \Vdash \phi$.

$\langle \overline{D} \rangle$: $\mathbf{M}, [d_0, d_1] \Vdash \langle \overline{D} \rangle \phi$ iff there exists d_2, d_3 s.t. $d_2 < d_0 < d_1 < d_3$ and $\mathbf{M}, [d_2, d_3] \Vdash \phi$.



Formal semantics of HS - contd'

$\langle L \rangle$: $\mathbf{M}, [d_0, d_1] \Vdash \langle L \rangle \phi$ iff there exists d_2, d_3 s.t. $d_1 < d_2 < d_3$ and $\mathbf{M}, [d_2, d_3] \Vdash \phi$.

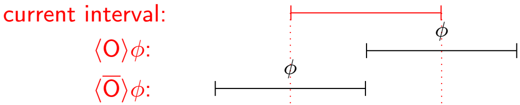
$\langle \overline{L} \rangle$: $\mathbf{M}, [d_0, d_1] \Vdash \langle \overline{L} \rangle \phi$ iff there exists d_2, d_3 s.t. $d_2 < d_3 < d_0$ and $\mathbf{M}, [d_2, d_3] \Vdash \phi$.

$\langle D \rangle$: $\mathbf{M}, [d_0, d_1] \Vdash \langle D \rangle \phi$ iff there exists d_2, d_3 s.t. $d_0 < d_2 < d_3 < d_1$ and $\mathbf{M}, [d_2, d_3] \Vdash \phi$.

$\langle \overline{D} \rangle$: $\mathbf{M}, [d_0, d_1] \Vdash \langle \overline{D} \rangle \phi$ iff there exists d_2, d_3 s.t. $d_2 < d_0 < d_1 < d_3$ and $\mathbf{M}, [d_2, d_3] \Vdash \phi$.

$\langle O \rangle$: $\mathbf{M}, [d_0, d_1] \Vdash \langle O \rangle \phi$ iff there exists d_2, d_3 s.t. $d_0 < d_2 < d_1 < d_3$ and $\mathbf{M}, [d_2, d_3] \Vdash \phi$.

$\langle \overline{O} \rangle$: $\mathbf{M}, [d_0, d_1] \Vdash \langle \overline{O} \rangle \phi$ iff there exists d_2, d_3 s.t. $d_2 < d_0 < d_3 < d_1$ and $\mathbf{M}, [d_2, d_3] \Vdash \phi$.



Formal semantics of HS - contd'

$\langle L \rangle$: $\mathbf{M}, [d_0, d_1] \Vdash \langle L \rangle \phi$ iff there exists d_2, d_3 s.t. $d_1 < d_2 < d_3$ and $\mathbf{M}, [d_2, d_3] \Vdash \phi$.

$\langle \overline{L} \rangle$: $\mathbf{M}, [d_0, d_1] \Vdash \langle \overline{L} \rangle \phi$ iff there exists d_2, d_3 s.t. $d_2 < d_3 < d_0$ and $\mathbf{M}, [d_2, d_3] \Vdash \phi$.

$\langle D \rangle$: $\mathbf{M}, [d_0, d_1] \Vdash \langle D \rangle \phi$ iff there exists d_2, d_3 s.t. $d_0 < d_2 < d_3 < d_1$ and $\mathbf{M}, [d_2, d_3] \Vdash \phi$.

$\langle \overline{D} \rangle$: $\mathbf{M}, [d_0, d_1] \Vdash \langle \overline{D} \rangle \phi$ iff there exists d_2, d_3 s.t. $d_2 < d_0 < d_1 < d_3$ and $\mathbf{M}, [d_2, d_3] \Vdash \phi$.

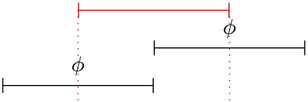
$\langle O \rangle$: $\mathbf{M}, [d_0, d_1] \Vdash \langle O \rangle \phi$ iff there exists d_2, d_3 s.t. $d_0 < d_2 < d_1 < d_3$ and $\mathbf{M}, [d_2, d_3] \Vdash \phi$.

$\langle \overline{O} \rangle$: $\mathbf{M}, [d_0, d_1] \Vdash \langle \overline{O} \rangle \phi$ iff there exists d_2, d_3 s.t. $d_2 < d_0 < d_3 < d_1$ and $\mathbf{M}, [d_2, d_3] \Vdash \phi$.

current interval:

$\langle O \rangle \phi$:

$\langle \overline{O} \rangle \phi$:



Thus, every HS-formula is interpreted in an interval model by a set of ordered pairs of points, i.e., by a binary relation.

So, no MSO decidability results, such as Rabin's theorem, apply.

Definabilities between interval modalities in HS

A useful new symbol is the modal constant π for point-intervals:

$$\mathbf{M}, [d_0, d_1] \Vdash \pi \text{ iff } d_0 = d_1.$$

Definable as $[B]\bot$ or $[E]\bot$, so only needed in weaker fragments of HS.

Definabilities between HS modalities in the strict / non-strict semantics:

- Later: $\langle L \rangle \varphi \equiv \langle A \rangle \langle A \rangle \varphi$ / $\langle L \rangle \varphi \equiv \langle A \rangle (\neg\pi \wedge \langle A \rangle \varphi)$;
- Before: $\langle \overline{L} \rangle \varphi \equiv \langle \overline{A} \rangle \langle \overline{A} \rangle \varphi$ / $\langle \overline{L} \rangle \varphi \equiv \langle \overline{A} \rangle (\neg\pi \wedge \langle \overline{A} \rangle \varphi)$;
- Overlaps on the right: $\langle O \rangle \varphi \equiv \langle E \rangle \langle \overline{B} \rangle \varphi$;
- Overlaps on the left: $\langle \overline{O} \rangle \varphi \equiv \langle B \rangle \langle \overline{E} \rangle \varphi$;
- During (strict sub-interval): $\langle D \rangle \varphi \equiv \langle B \rangle \langle E \rangle \varphi \equiv \langle E \rangle \langle B \rangle \varphi$.

  strict super-interval: $\langle \overline{D} \rangle \varphi \equiv \langle \overline{B} \rangle \langle \overline{E} \rangle \varphi \equiv \langle \overline{E} \rangle \langle \overline{B} \rangle \varphi$.
- Right neighbour (in non-strict semantics): $\langle A \rangle \varphi \equiv \langle E \rangle (\pi \wedge \langle \overline{B} \rangle \varphi)$.
- Left neighbour (in non-strict semantics): $\langle \overline{A} \rangle \varphi \equiv \langle B \rangle (\pi \wedge \langle \overline{E} \rangle \varphi)$.

Minimal languages for Halpern-Shoham's interval logic HS

Thus, in the non-strict semantics, it suffices to choose as primitive the modalities $\langle \mathsf{B} \rangle, \langle \mathsf{E} \rangle, \langle \overline{\mathsf{B}} \rangle, \langle \overline{\mathsf{E}} \rangle$ corresponding to the relations *begins, ends*, , and their inverses; the other modalities then become definable.

So, the formulas of Halpern-Shoham's logic, can be defined as:

$$\phi ::= p \mid \neg\phi \mid \phi \wedge \psi \mid \langle \mathsf{B} \rangle\phi \mid \langle \mathsf{E} \rangle\phi \mid \langle \overline{\mathsf{B}} \rangle\phi \mid \langle \overline{\mathsf{E}} \rangle\phi.$$

In the strict semantics, the modalities $\langle A \rangle$ and $\langle \overline{A} \rangle$ over the right and left neighbourhood relations must be added.

Hereafter, I will focus on the strict semantics.

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Comparing the expressiveness of interval logics and FO

Every interval model $\mathbf{M} = \langle D, <, V \rangle$ can be regarded as a structure for $\mathrm{FO}[<]$ with infinitely many binary predicates $\{P_n\}_{n \in \mathbb{N}}$, by interpreting them with the valuations of the atomic propositions.

By standard translation, HS is thus embedded as a fragment of FO. Actually, as a fragment of $\mathrm{FO}^3[<]$.

Theorem (Venema'1991)

CDT *is expressively complete for* $\mathrm{FO}^3[<]$.

Question: what part of HS can be translated to $\mathrm{FO}^2[<]$?

Theorem (Goranko, Montanari, Sciavicco, LFCS'2007, APAL'2009)

$\mathsf{PNL}^{\pi+} = \mathsf{A}\overline{\mathsf{A}} + \pi$ *is expressively complete for* $\mathrm{FO}^2[<]$.

Therefore, $\mathsf{PNL}^{\pi+}$ is the largest fragment of HS that can be translated (with the non-strict semantics) into $\mathrm{FO}^2[<]$.

On the relative expressiveness of the fragments of HS

Every subset of the 12 Allen's relations (excl. the equality) defines a fragment of HS. Thus, there are $2^{12} = 4096$ fragments of HS.

However, some of them are equally expressible due to inter-definabilities.

Expressiveness classification programme:
classify all fragments of HS with respect to their expressiveness, relative to natural classes of interval models.

Proving $L_1 \preceq L_2$ is constructive, by means of a truth-preserving translation from L_1 into L_2.

Disproving $L_1 \preceq L_2$ requires model-theoretic methods, typically based on invariance/non-invariance under suitable bisimulations.

Using these, plus computer aided check of reductions, a minimal complete set of identities has been identified.

Summary of the relative expressiveness of fragments of HS

Theorem (Della Monica, Montanari, Goranko, Sciavicco; IJCAI' 2011)

In the strict semantics over the class of all linear orders there are exactly 1347 expressively distinct fragments of HS.

Theorem (Aceto, Della Monica, Ingólfsdóttir, Montanari, Sciavicco TIME'2013)

In the strict semantics over the class of dense linear orders there are exactly 966 expressively distinct fragments of HS.

Full details in:

Aceto, Della Monica, Ingólfsdóttir, Montanari, Goranko, Sciavicco: Complete classification of the expressiveness of interval logics of Allen's relations. The general and the dense cases. *Acta Informatica*, Vol. 53(3), 2016, p 207-246.

Method: establish a compete set of definabilities between strings of modalities.

The classifications on the classes of finite linear orders and discrete linear orders nearly completed (modulo some open cases with $\langle O \rangle$ and $\langle \overline{O} \rangle$) in: (*Aceto, Della Monica, Ingólfsdóttir, Montanari, Sciavicco; JELIA'2014*). Reportedly, completed recently by Della Monica.

Open question: expressiveness on the class of all ordinals.

Minimal complete set of definabilities among modalities on the class of all linear orders

J. Halpern and Y. Shoham: A propositional modal logic of time intervals. *Journal of the ACM*, 1991

$$\langle L \rangle \sqsubseteq A \quad \langle L \rangle p \equiv \langle A \rangle \langle A \rangle p$$
$$\langle D \rangle \sqsubseteq BE \quad \langle D \rangle p \equiv \langle B \rangle \langle E \rangle p$$
$$\langle O \rangle \sqsubseteq \overline{B}E \quad \langle O \rangle p \equiv \langle E \rangle \langle \overline{B} \rangle p$$
$$\langle L \rangle \sqsubseteq \overline{B}E \quad \langle L \rangle p \equiv \langle \overline{B} \rangle [E] \langle \overline{B} \rangle \langle E \rangle p$$

D. Della Monica, V. Goranko, A. Montanari, and G. Sciavicco: Expressiveness of the Interval Logics of Allen's Relations on the Class of all Linear Orders: Complete Classification. *IJCAI'2011*

Technique for proving completeness: prove all non-definabilities by using suitable bisimulations.

Minimal complete set of definabilities among modalities on the class of dense linear orders

$$
\begin{array}{lcllcl}
\langle \mathsf{L} \rangle & \sqsubseteq & \mathsf{A} & \langle \mathsf{L} \rangle p & \equiv & \langle \mathsf{A} \rangle \langle \mathsf{A} \rangle p \\
\langle \mathsf{D} \rangle & \sqsubseteq & \mathsf{BE} & \langle \mathsf{D} \rangle p & \equiv & \langle \mathsf{B} \rangle \langle \mathsf{E} \rangle p \\
\langle \mathsf{O} \rangle & \sqsubseteq & \overline{\mathsf{B}}\mathsf{E} & \langle \mathsf{O} \rangle p & \equiv & \langle \mathsf{E} \rangle \langle \overline{\mathsf{B}} \rangle p \\
\langle \mathsf{L} \rangle & \sqsubseteq & \overline{\mathsf{B}}\mathsf{E} & \langle \mathsf{L} \rangle p & \equiv & \langle \overline{\mathsf{B}} \rangle [\mathsf{E}] \langle \overline{\mathsf{B}} \rangle \langle \mathsf{E} \rangle p \\
\langle \mathsf{L} \rangle & \sqsubseteq & \mathsf{DO} & \langle \mathsf{L} \rangle p & \equiv & \langle \mathsf{O} \rangle (\langle \mathsf{O} \rangle \top \wedge [\mathsf{O}](\langle \mathsf{O} \rangle p \vee \langle \mathsf{D} \rangle p \vee \langle \mathsf{D} \rangle \langle \mathsf{O} \rangle p)) \\
\langle \mathsf{L} \rangle & \sqsubseteq & \overline{\mathsf{B}}\mathsf{D} & \langle \mathsf{L} \rangle p & \equiv & \langle \overline{\mathsf{B}} \rangle [\mathsf{D}] \langle \overline{\mathsf{B}} \rangle \langle \mathsf{D} \rangle \langle \overline{\mathsf{B}} \rangle p \\
\langle \mathsf{L} \rangle & \sqsubseteq & \mathsf{EO} & \langle \mathsf{L} \rangle p & \equiv & \langle \mathsf{O} \rangle [\mathsf{E}] \langle \mathsf{O} \rangle \langle \mathsf{O} \rangle p \\
\langle \mathsf{L} \rangle & \sqsubseteq & \mathsf{BO} & \langle \mathsf{L} \rangle p & \equiv & \langle \mathsf{O} \rangle (\langle \mathsf{O} \rangle \top \wedge [\mathsf{O}] \langle \mathsf{B} \rangle \langle \mathsf{O} \rangle \langle \mathsf{O} \rangle p) \\
\langle \mathsf{L} \rangle & \sqsubseteq & \overline{\mathsf{L}}\mathsf{O} & \langle \mathsf{L} \rangle p & \equiv & \langle \mathsf{O} \rangle [\mathsf{O}] [\overline{\mathsf{L}}] \langle \mathsf{O} \rangle \langle \mathsf{O} \rangle p
\end{array}
$$

L. Aceto, D. Della Monica, A. Ingolfsdottir, A. Montanari, and G. Sciavicco: A complete classification of the expressiveness of interval logics of Allen's relations over dense linear orders. *TIME'2013*

Complete set of definabilities among modalities on the class of discrete linear orders

$$
\begin{array}{lllll}
\langle \mathrm{L} \rangle & \sqsubseteq & \mathrm{A} & \langle \mathrm{L} \rangle p & \equiv \langle \mathrm{A} \rangle \langle \mathrm{A} \rangle p \\
\langle \mathrm{D} \rangle & \sqsubseteq & \mathrm{BE} & \langle \mathrm{D} \rangle p & \equiv \langle \mathrm{B} \rangle \langle \mathrm{E} \rangle p \\
\langle \mathrm{O} \rangle & \sqsubseteq & \overline{\mathrm{B}}\mathrm{E} & \langle \mathrm{O} \rangle p & \equiv \langle \mathrm{E} \rangle \langle \overline{\mathrm{B}} \rangle p \\
\langle \mathrm{L} \rangle & \sqsubseteq & \overline{\mathrm{B}}\mathrm{E} & \langle \mathrm{L} \rangle p & \equiv \langle \overline{\mathrm{B}} \rangle [\mathrm{E}] \langle \overline{\mathrm{B}} \rangle \langle \mathrm{E} \rangle p \\
\langle \mathrm{A} \rangle & \sqsubseteq & \overline{\mathrm{B}}\mathrm{E} & \langle \mathrm{A} \rangle p & \equiv \varphi(p) \vee \langle \mathrm{E} \rangle \varphi(p)
\end{array}
$$

where $\varphi(p) = [\mathrm{E}]\bot \wedge \langle \overline{\mathrm{B}} \rangle ([\mathrm{E}][\mathrm{E}]\bot \wedge \langle \mathrm{E} \rangle (p \vee \langle \overline{\mathrm{B}} \rangle p))$

L. Aceto, D. Della Monica, A. Ingolfsdottir, A. Montanari, and G. Sciavicco: On the expressiveness of the interval logic of Allen's relations over finite and discrete linear orders. *JELIA'2014*

Open questions: $\langle \mathrm{O} \rangle \sqsubseteq$? on discrete linear orders? On ordinals?

Outline

Undecidability in interval logics: the initial bad news

Theorem [Halpern and Shoham'91] The validity in HS, in the non-strict semantics, over any class of ordered structures containing at least one with an infinitely ascending sequence is r.e.-hard.

Thus, in particular, HS is undecidable over the classes of all (non-strict) models, all linear models, all discrete linear models, all dense linear models, $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{R}$, etc.

Proof idea: reduction from the non-halting problem for Turing machines to testing satisfiability in HS.

Undecidability in interval logics: it can be worse...

Theorem [Halpern and Shoham'91] The validity in HS over any class of Dedekind complete ordered structures containing at least one with an infinitely ascending sequence is Π^1_1-hard.

In particular, the validity in HS over any of the orderings of the natural numbers, integers, or reals is not recursively axiomatizable.

Proof idea: reduction from the problem of existence of a computation of a given non-deterministic Turing machine that enters the start state infinitely often to testing satisfiability in HS.

... and even worse

Undecidability occurs even without existence of infinitely ascending sequences. A class of ordered structures has unboundedly ascending sequences if for every *n* there is a structure in the class with an ascending sequence of length at least *n*.

Theorem [Halpern and Shoham'91] The validity problem in HS interpreted over any class of Dedekind complete ordered structures having unboundedly ascending sequences is co-r.e. hard.

In particular, satisfiability of HS formulae in the finite is r.e. hard.

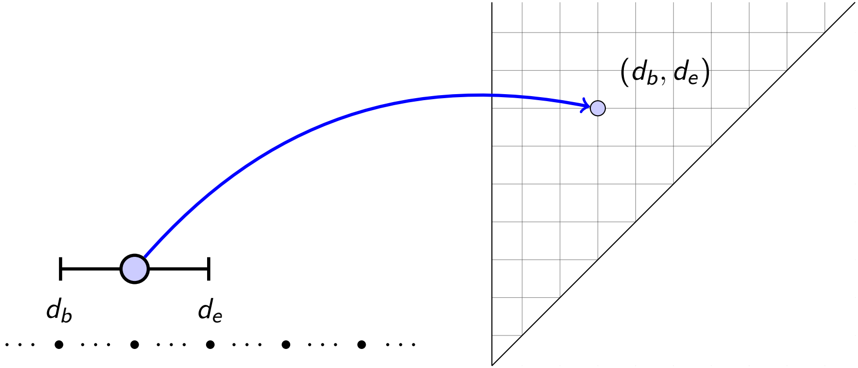
Proof idea: reduction from the halting problem for Turing machines to testing satisfiability in HS.

In all these cases, recursive axiomatizations cannot exist.

Some sharper follow-up results: by Lodaya and others.

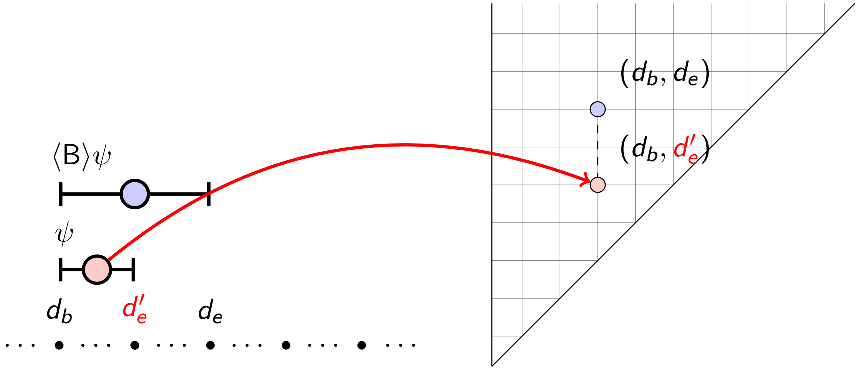
Stronger and more recent results: based on planar interpretation of interval structures and reduction from tiling problems.

Geometric interpretation of interval structures: intervals



Every interval on $\mathbb{N}$ can be represented by a point in the second octant (in general, in the half plane $y \geq x$)

Geometric interpretation of interval relations



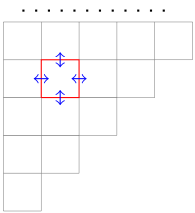
$d_b < d'_e < d_e$

Every interval relation has a spatial counterpart

The Octant Tiling Problem

The 2nd octant of $\mathbb{Z} \times \mathbb{Z}$:

$$\mathcal{O} = \{(i,j) : i,j \in \mathbb{N} \land 0 \leq i \leq j\}$$



A natural interpretation of intervals on $\mathbb{N}$ into $\mathcal{O}$.

The Octant Tiling Problem: can a given finite set of tile types $\mathcal{T} = \{t_1, \ldots, t_k\}$ tile $\mathcal{O}$ while respecting the color constraints?

Fact: the Octant Tiling Problem is undecidable.

(By reduction from the the tiling problem for $\mathbb{N} \times \mathbb{N}$.)

A sample result using the Octant Tiling problem: undecidability of the logic O over (discrete) linear orderings

Theorem (Bresolin, Della Monica, Goranko, Montanari, Sciavicco TIME'2009)

The satisfiability problem for the logic O *is undecidable over any class of discrete linear orderings that contains at least one linear ordering with an infinite ascending sequence.*

Proof: for every finite set of tiles $\mathcal{T}$ we build a formula $\phi_{\mathcal{T}} \in \mathrm{O}$ such that

$\phi_{\mathcal{T}}$ is satisfiable in a discrete linear ordering

iff

$\mathcal{T}$ can tile the 2nd octant.

O and $\overline{\mathrm{O}}$ were the first uni-modal fragments of HS proved undecidable over the class of discrete orderings.

More recent results on undecidability of interval logics

Using variations of the Octant Tiling Problem encoding:

Theorem (Bresolin, Della Monica, G., Montanari, Sciavicco; TIME'2011)

The satisfiability problem for each of the HS *fragments* O, $\overline{\mathrm{O}}$, AD, A$\overline{\mathrm{D}}$, $\overline{\mathrm{A}}$D, $\overline{\mathrm{A}}\overline{\mathrm{D}}$, BE, B$\overline{\mathrm{E}}$, $\overline{\mathrm{B}}$E, $\overline{\mathrm{B}}\overline{\mathrm{E}}$, *is undecidable in any class of linear orders that, for each $n > 0$, contains a linear order with length greater than n.*

Other, unexpected results: *undecidability of D and of $\overline{D}$ over $\mathbb{N}$ for proper or strict subintervals (with either semantics) and of $D\overline{D}$ over all linear orderings, with non-strict semantics.*

- J. Marcinkowski and J. Michaliszyn, *The Ultimate Undecidability Result for the Halpern-Shoham Logic*, LICS 2011

Proof by reduction from the halting problem for two-counter automata.

Open questions: decidability of D or $\overline{D}$ on all linear orders? Decidability of $D\overline{D}$ over all linear orderings, with strict semantics.

A recent survey in:

- D. Bresolin, D. Della Monica, V. Goranko, A. Montanari, and G. Sciavicco, The dark side of Interval Temporal Logics: marking the undecidability border, Annals of Mathematics and Artificial Intelligence, 2014

Outline

Decidability of interval logics by semantic restrictions

Most decidability results in interval logics are based either on weak expressiveness or on some semantic restrictions reducing the interval-based semantics to point-based, e.g.:

- Homogeneity: an atomic proposition is true of an interval iff it is true of every subinterval / every point in that interval.

  In Duration Calculus: true at 'almost every point'.

- Locality: an atomic proposition is true of an interval if and only if it is true of its starting point.

- Incomplete interval structures, not containing all intervals, e.g., *split structures*, where every interval can be chopped in a unique way (thus creating a tree-like subinterval structure).

Easy decidable cases of fragments of HS: $B\overline{B}$ and $E\overline{E}$. They essentially are point-based, as one end-point of the current interval remains fixed.

Decidable interval logics with genuine interval semantics

So, are there any non-trivially decidable fragments of HS? Yes!

The border between decidability and undecidability turns out quite subtle.

Classifying the decidable and undecidable fragments of HS has been the most challenging research direction in the area.

Now, nearly completed, but with a few still open problems.

Decidability of interval neighborhood logics via reduction to $FO^2[<]$

- Recall that the fragment $A\overline{A}\pi$, aka $PNL^{\pi+}$, is expressively complete for $FO^2[<]$.
- Satisfiability in $FO^2[<]$ was first proved decidable (NEXPTIME-complete) by Martin Otto [JSL'2001] on:
  - The class of all linear orders;
  - The class of all linear well-orders;
  - The orders of naturals $\mathbb{N}$ and integers $\mathbb{Z}$.

  Proof based on an elaborated model-theoretic argument, analyzing the types of elements and pairs in models of $FO^2[<]$,
- Hence the decidability of $A\overline{A}\pi$ on each of these classes.
- Decidability of $A\overline{A}$ on the strict semantics now follows easily.

Decidability of interval logics via tableaux

Many recent decidability results for fragments of HS were established by using tableau-based decision procedures. Some generic features:

- Often all standard models are infinite, so the tableau constructions require special loop control mechanisms to guarantee termination.
- Usually open saturated tableaux do not produce directly models, but only finite pseudo-models, which are then expanded to (possibly infinite) standard models.
- Thus, there are three stages in proving satisfiability of a formula in a given class of interval structures:
- open tableau $\Rightarrow$ pseudo-model $\Rightarrow$ standard model.
- Both transitions are provably successful and constructive, thus reducing the task of testing satisfiability to the construction of an open saturated tableau for the formula.

The strongest recent decidability results have been obtained by a more streamlined argument, proving directly small pseudo-model property.

Decidability of interval logics via tableaux: partial summary

First results:

A:

- D. Bresolin and A. Montanari: A Tableau-based Decision Procedure for Right Propositional Neighborhood Logic, *TABLEAUX 2005, JAR'2007.*

$A\overline{A}$:

- D. Bresolin, A. Montanari, and P. Sala: An Optimal Tableau-based Decision Algorithm for Propositional Neighborhood Logic, *STACS 2007*

D on dense orderings:

- D. Bresolin, V. Goranko, A. Montanari, and P. Sala: Tableau systems for logics of subinterval structures over dense orderings, *Tableaux'2007, J. Logic. and Comput. 2010*

Decidability of interval logics: partial summary cont'd

Reflexive $D\overline{D}$ on finite linear orders.

A. Montanari, I. Pratt-Hartmann, P. Sala: Decidability of the Logics of the Reflexive Sub-interval and Super-interval Relations over Finite Linear Orders. *TIME 2010*.

$A\overline{A}$ over all, dense, and discrete linear orders:

D. Bresolin, A. Montanari, P. Sala, and G. Sciavicco: Optimal Tableau Systems for Propositional Neighborhood Logic over All, Dense, and Discrete Linear Orders, *TABLEAUX 2011*.

$A\overline{A}$ over the reals:

A. Montanari and P. Sala: An optimal tableau system for the logic of temporal neighborhood over the reals, *TIME 2012*.

Maximal decidable fragments: summary and references

Over finite linear orders: $AB\overline{B}\overline{A}$ and $AE\overline{E}\overline{A}$

A. Montanari, G. Puppis, and P. Sala: Maximal decidable fragments of Halpern and Shoham's modal logic of intervals, ICALP 2010

Over $\mathbb{N}$ and strongly discrete orders: $AB\overline{B}\overline{L}$ and $AE\overline{E}\overline{L}$

D. Bresolin, D. Della Monica, A. Montanari, P. Sala, G. Sciavicco: Interval temporal logics over strongly discrete linear orders: Expressiveness and complexity. *Theor. Comput. Sci. 560: 269-291 (2014)*

Over $\mathbb{Q}$ and all dense linear orders: $AB\overline{B}\overline{A}$, $AE\overline{E}\overline{A}$ and $B\overline{B}D\overline{D}L\overline{L}$

D. Bresolin, D. Della Monica, A. Montanari, P. Sala, G. Sciavicco: On the Complexity of Fragments of the Modal Logic of Allen's Relations over Dense Structures. *LATA 2014*

Over all, dense, and strongly discrete linear orders: $AB\overline{B}\overline{L}$

D. Bresolin, A. Montanari, P. Sala, and G. Sciavicco: What's decidable about Halpern and Shoham's interval logic? The maximal fragment $AB\overline{B}\overline{L}$, *LICS 2011*

Decidable fragments of HS on the class of finite orders: a chart

Bresolin, Della Monica, Montanari, Sala, Sciavicco: Interval Temporal Logics over Finite Linear Orders: the Complete Picture *Proc. of ECAI'2012*

Of the 1347 expressively different fragments of HS, only the following 35 and their symmetric versions are decidable over the class of finite linear orders:

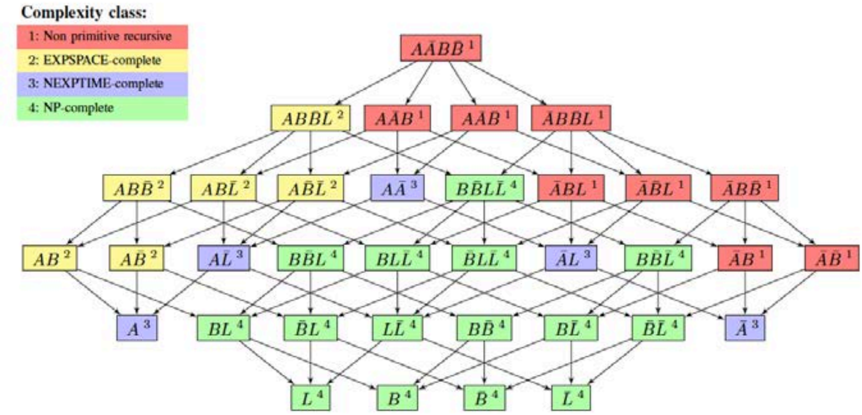


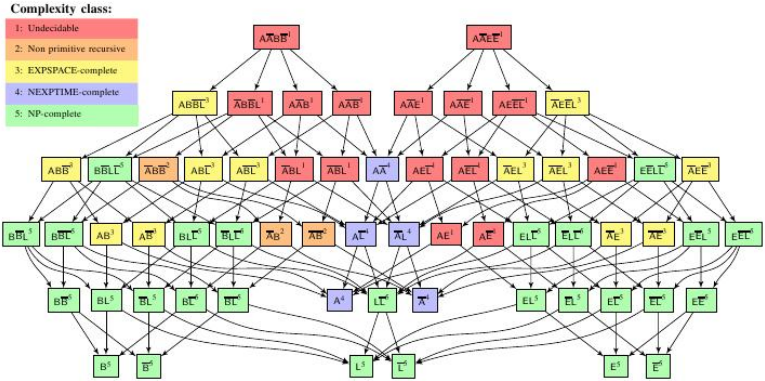
Figure 2. Hasse diagram of all and only decidable fragments of HS over finite linear orders.

nko

Decidable fragments of HS on strongly discrete linear orders: a chart

D. Bresolin, D. Della Monica, A. Montanari, P. Sala, G. Sciavicco: Interval temporal logics over strongly discrete linear orders: Expressiveness and complexity. *Theor. Comput. Sci. 560: 269-291 (2014)*

Chart of the maximal fragments of HS, with sub-fragments decidable over $\mathbb{N}$ and over all l strongly discrete linear orders:



Outline

Introduction. Interval structures and relations

Interval temporal logics

Halpern-Shoham's logic HS and its fragments

Relative expressiveness of the fragments of HS

Undecidability

Decidability

Representation theorems and axiomatic systems

Interval relational structures
Interval frames and representation theorems

Interval relational structure: an interval structure enriched with one or more designated interval relations. For instance:

- sub-interval structures,
- begin-end structures,
- left-right neighbourhood structures,
- chop structures, etc.

Interval frame: an abstract structure with similarity type of an interval relational structure, e.g., sub-interval frames, begin-end frames, neighbourhood frames, chop frames, etc.

Abstract characterizations for interval relational structures

Let $\mathcal{R} = \{\mathrm{R}_1, \ldots, \mathrm{R}_\mathrm{k}\}$ be an interval relational type.

The abstract characterization problem for $\mathcal{R}$:

> *Identify a (first-order) theory, the models of which are isomorphic to interval relational structures of type $\mathcal{R}$.*

I.e., establish a representation theorem for $\mathcal{R}$.

Weaker problem:

> *Identify the (FO) theory of the class of interval relational frames that are isomorphically embeddable into interval relational structures of type $\mathcal{R}$.*

(Incomplete) summary of representation theorems for classes of interval structures

- van Benthem, 1983: the subinterval-precedence structure over the rationals.
- Allen & Hayes, 1985: the meet-structure over the rationals.
- Ladkin, 1987: point-based structures with a quaternary relation, encoding the meet of two intervals.
- Venema, 1988: begin-end structures.
- Goranko, Montanari, Sciavicco, 2003: neighbourhood structures on various classes with strict and non-strict semantics.
- Bresolin and Goranko, 2005 (unpubl.): some sub-interval structures, incl. over $\mathbb{N}$ and $\mathbb{Z}$.
- Conradie & Coetzee, 2008 (unpubl.): overlap-after structures.

Some open cases: most natural classes of: subinterval and superinterval structures; of overlap structures, etc.

Axiomatic systems for interval logics: summary

Few complete axiomatic systems for interval logics have been obtained. Mostly separate results, with the exception of PNL ($=A\overline{A}$):

- Moszkowski: complete axiomatizations of variants of ITL.
- Venema'90 axiomatizations of HS and CDT over all linear orders, using unorthodox inference rules.
- Shehtman and Shapirovsky: logics for Minkowski spacetime, related to some interval logics on dense orders.
- Axiomatic systems for PNL for strict and non-strict semantics on the most important classes of interval structures on linear orders:

V. Goranko, A. Montanari, and G. Sciavicco: Propositional Interval Neighborhood Logics, *Journal of Universal Computer Science, vol 9, No. 9 (2003), pp. 1137-1167.*

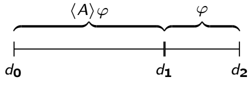
For a survey, see also:

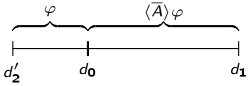
V. Goranko, A. Montanari, and G. Sciavicco: A Road Map on Interval Temporal Logics and Duration Calculi, *Journal of Applied Non-classical Logics, vol. 14, No. 1-2, 2004, pp.11-56.*

Propositional interval neighbourhood logics (PNL) revisited

Here I discuss PNL = $A\overline{A}$ with strict semantics only.

Recall the semantics:

after : 

before : 

Difference modality definable:

$$\langle\neq\rangle\varphi \equiv \langle\overline{\mathsf{A}}\rangle\langle\overline{\mathsf{A}}\rangle\langle\mathsf{A}\rangle\varphi \vee \langle\overline{\mathsf{A}}\rangle\langle\mathsf{A}\rangle\langle\mathsf{A}\rangle\varphi \vee \langle\mathsf{A}\rangle\langle\overline{\mathsf{A}}\rangle\langle\overline{\mathsf{A}}\rangle\varphi \vee \langle\mathsf{A}\rangle\langle\mathsf{A}\rangle\langle\overline{\mathsf{A}}\rangle\varphi.$$

Axioms for the strict propositional neighbourhood logics

Theorem (Goranko, Montanari, Sciavicco; JUCS'2003)

The following axiomatic system Ax_{PNL} is sound and complete for the logic PNL *of all linear orders:*

- Ax_{PNL}-1 *enough propositional tautologies,*
- Ax_{PNL}-2 *the K axioms for $[A]$ and $[\overline{A}]$,*
- Ax_{PNL}-3 $p \to [A]\langle\overline{A}\rangle p$ *and its inverse,*
- Ax_{PNL}-4 $\langle A\rangle\langle\overline{A}\rangle p \to [A]\langle\overline{A}\rangle p$ *and its inverse,*
- Ax_{PNL}-5 $(\langle\overline{A}\rangle\langle\overline{A}\rangle\top \wedge \langle A\rangle\langle\overline{A}\rangle p) \to p \vee \langle\overline{A}\rangle\langle A\rangle\langle A\rangle p \vee \langle\overline{A}\rangle\langle\overline{A}\rangle\langle A\rangle p$ *and its inverse,*
- Ax_{PNL}-6 $\langle A\rangle\langle A\rangle\langle A\rangle p \to \langle A\rangle\langle A\rangle p$ *and its inverse.*

(A side remark: the extension of **K** with the last axiom above, $\Diamond\Diamond\Diamond p \to \Diamond\Diamond p$, is not known yet to be decidable or not.)

Some PNL-definable classes of neighbourhood structures

- The formula (SPNL^{der})

$$(\langle A\rangle\langle A\rangle p \rightarrow \langle A\rangle\langle A\rangle\langle A\rangle p) \wedge (\langle A\rangle[A]p \rightarrow \langle A\rangle\langle A\rangle[A]p)$$

  together with its left inverse (SPNL^{del})

$$(\langle \overline{A}\rangle\langle \overline{A}\rangle p \rightarrow \langle \overline{A}\rangle\langle \overline{A}\rangle\langle \overline{A}\rangle p) \wedge (\langle \overline{A}\rangle[\overline{A}]p \rightarrow \langle \overline{A}\rangle\langle \overline{A}\rangle[\overline{A}]p)$$

  defines the class of dense structures (plus the 2-element structure);

- The formula (SPNL^{dir})

  $[A](p \wedge [\overline{A}]\neg p \wedge [A]p) \rightarrow [\overline{A}][\overline{A}]\langle A\rangle((\langle A\rangle\neg p \wedge [A][A]p) \vee (\langle A\rangle\top \wedge [A][A]\bot))$

  together with its inverse (SPNL^{dil}) defines the class of discrete structures;

- The formula (SPNL^{dc})

$$\langle A\rangle\langle A\rangle[\overline{A}]p \wedge \langle A\rangle[A]\neg[\overline{A}]p \rightarrow \langle A\rangle(\langle A\rangle[\overline{A}][\overline{A}]p \wedge [A]\langle A\rangle\neg[\overline{A}]p)$$

  defines the class of Dedekind complete structures.

Some complete axiomatic extensions of PNL

Theorem (Goranko, Montanari, Sciavicco; JUCS'2003)

Each of the following axiomatic extensions of PNL *is sound and complete for the respective class of neighbourhood structures:*

- $Ax^{de}_{\mathsf{PNL}} = Ax_{\mathsf{PNL}} + \mathsf{SPNL}^{der} + \mathsf{SPNL}^{del}$
- $Ax^{di}_{\mathsf{PNL}} = Ax_{\mathsf{PNL}} + \mathsf{SPNL}^{dir} + \mathsf{SPNL}^{dil}$
- $Ax^{dc}_{\mathsf{PNL}} = Ax_{\mathsf{PNL}} + \mathsf{SPNL}^{dc}$

Similar results for the non-strict semantics.

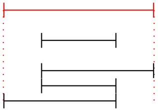
Logics of subintervals D revisited

Recall the generic semantics: $\mathbf{M}, [d_0, d_1] \Vdash \langle \mathrm{D} \rangle \phi$ iff there exists a (any/proper/strict) subinterval $[d_2, d_3]$ of $[d_0, d_1]$ such that $\mathbf{M}, [d_2, d_3] \Vdash \phi$.

current interval:

strict subinterval:

proper subintervals:



The irreflexive (proper or strict) **D** is quite expressive. For instance:

• combinatorial relationships between the number of subintervals and the depth of an interval can be expressed, such as:

$$\bigwedge_{i=1}^{d(n)} \langle \mathrm{D} \rangle \left(p_i \wedge \bigwedge_{j \neq i} \neg p_j \right) \rightarrow \langle \mathrm{D} \rangle^n \top$$

for a large enough $d(n)$.

• special properties of the models, e.g.: the formula

$$\langle \mathrm{D} \rangle \langle \mathrm{D} \rangle \top \wedge [\mathrm{D}](\langle \mathrm{D} \rangle \top \rightarrow \langle \mathrm{D} \rangle \langle \mathrm{D} \rangle \top \wedge \langle \mathrm{D} \rangle [\mathrm{D}] \bot)$$

for proper subinterval relation has no discrete or dense models in the strict semantics, but is satisfiable in the Cantor space over $\mathbb{R}$.

On axiomatic systems for subinterval logics

- The logic D^{ref} of reflexive subintervals on all linear orders is **S4**.

  J. van Benthem, The Logic of Time (2nd Edition), Kluwer, 1991

- The logic D^{ref} on $\mathbb{N}$ strictly contains **S4Grz**.

  A. Montanari, I. Pratt-Hartmann, P. Sala: Decidability of the Logics of the Reflexive Sub-interval and Super-interval Relations over Finite Linear Orders. *TIME 2010*.

- The logic D^{str} of strict subintervals on dense linear orders is **KD4** + (the 2-density axiom) $\langle D\rangle p_1 \wedge \langle D\rangle p_2 \rightarrow \langle D\rangle(\langle D\rangle p_1 \wedge \langle D\rangle p_2)$. That logic is decidable and PSPACE-complete.

  I. Shapirowsky and V. Shehtman. Chronological future modality in Minkowski spacetime. *Advances in Modal Logic*, volume 4, London, 2003.

- Likewise, the interval logic D^{wa} of 'weakly after' (overlaps or later) on dense linear orders is $\mathbf{S4.2} = \mathbf{S4} + \Diamond\Box p \rightarrow \Box\Diamond p$.

Some open problems on axiomatic systems

Complete axiomatizations unknown for:

- the logic D^{ref} of subintervals on most natural subclasses of linear orders.
- the logic D^{str} of strict subintervals on most natural classes of linear orders.
- the logic D^{pr} of proper subintervals on any natural classes of linear orders.
- each of the above, for the logic of superintervals $\overline{D}$ and for the fragment $D\overline{D}$.
- the logics of Overlap.
- all maximal decidable fragments of HS mentioned earlier.
- and many more...

Summary and perspectives

This talk outlined several major topics on interval logics, and more specifically, on the class of fragments of Halpern-Shoham's logic HS.

The main research agenda so far: classifications of expressiveness and (un)decidability of fragments of HS.

Some little explored yet topics:

- Representation theorems and axiomatic systems
- Metric interval logics
- Model checking of Interval logics

In summary:
the land of interval logics is rich with challenges for modal logicians.

THE END