The length of distinguishing modal formulae

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Setting

Our setting: basic modal logic.

$$\varphi ::= p \mid \neg\varphi \mid (\varphi \vee \varphi) \mid (\varphi \wedge \varphi) \mid \Box\varphi \mid \Diamond\varphi$$

The question

Our question: given pointed models M_1, w_1 and M_2, w_2, what is the length of the shortest formula that distinguishes between M_1, w_1 and M_2, w_2?

- Length is number of symbols in the formula.
- Worst case.
- Compared to $|M_1| + |M_2|$.

Exponential bound

Theorem

If M_1, w_1 and M_2, w_2 are distinguishable, then there is a formula of exponential length (w.r.t. $|M_1| + |M_2|$) that distinguishes between them.

- Very unsurprising.
- More or less already known.
- But: we couldn't find anyone explicitly stating it.

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The exponential bound is tight.

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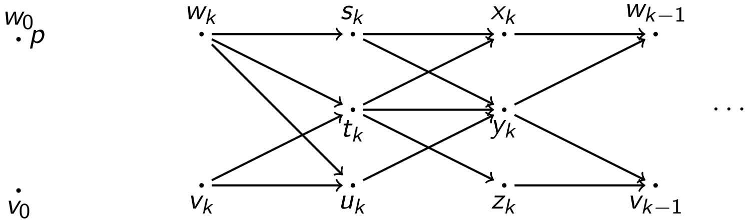
There is a sequence $\{M_k \mid k \in \mathbb{N}\}$ of models such that:

- *for every $k \in \mathbb{N}$, M_k, w_k and M_k, v_k are distinguishable,*
- *the size of M_k grows linearly with k,*
- *the length of the smallest formula that distinguishes between M_k, w_k and M_k, v_k grows exponentially with k*

Proof: by explicitly constructing the models.

Tight bound: proof



(a) M_0 (b) M_k for $k > 0$

- Suppose $M_k, w_k \models \varphi$ and $M_k, v_k \not\models \varphi$.
- Successors of v_k: subset of successors of w_k.
- Therefore: $\varphi = \Diamond\psi$, where $s_k \models \psi$, $t_k \not\models \psi$ and $u_k \not\models \psi$.

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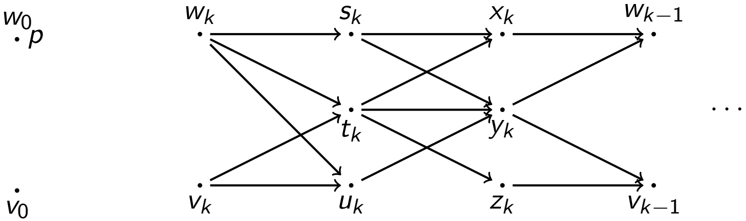

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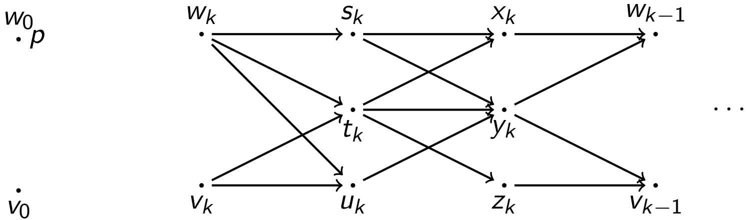

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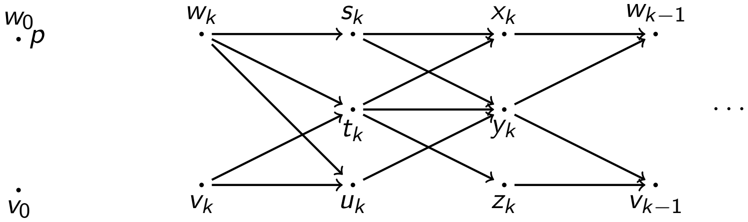

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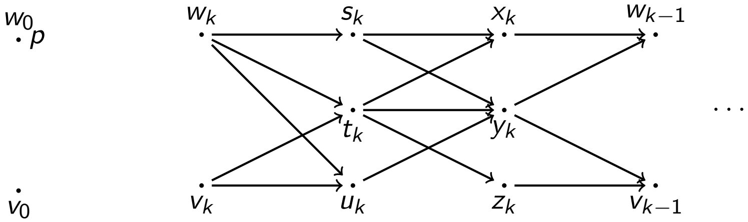

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- Successors of s_k: superset of successors of u_k. Therefore: ψ has subformula $\Diamond\xi$ where $x_x \models \xi$, $y_k \not\models \xi$.
- $\psi = \Box\chi \wedge \Diamond\xi$.

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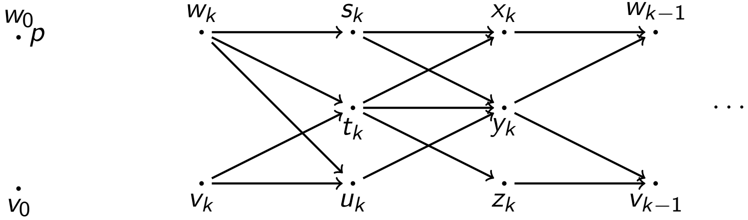

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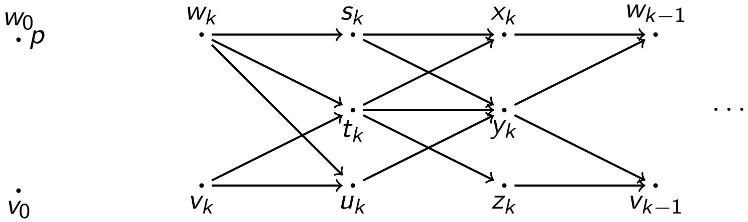

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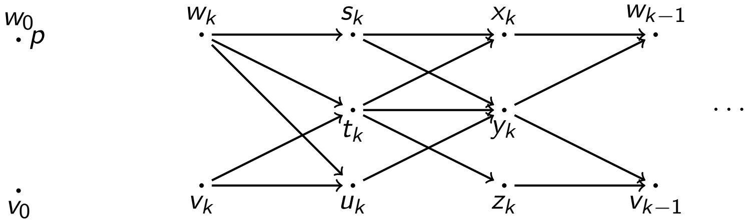

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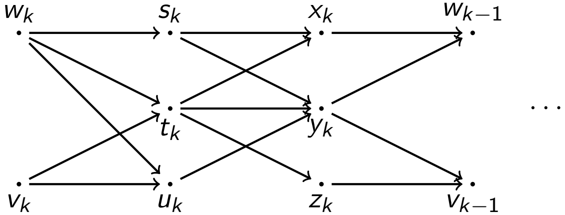
Tight bound: proof (III)

(a) M_0



(b) M_k for $k > 0$

- Recall: $\varphi = \Diamond(\Box\chi \wedge \Diamond\xi)$, with $x_k \models \chi, y_k \models \chi, z_k \models \chi$ and $x_k \models \xi$, $y_k \not\models \xi$.
- Distinguishing between x_k, y_k and/or x_k: at least as difficult as between w_{k-1} and v_{k-1}.
- Therefore: φ is at least twice as long as shortest formula distinguishing between w_{k-1} and v_{k-1}.

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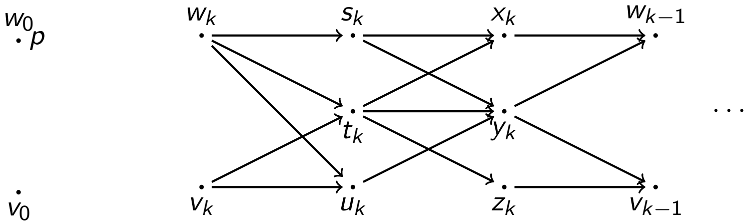

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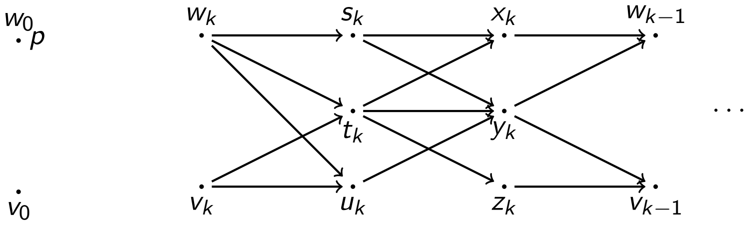

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Concluding remarks

- Worst-case formula length is exponential. But precise bound still unknown.
- Results generalize to other logics: e.g. multi-agent modal logic, tense logic, CTL, CTL*.
- Not known whether exponential bound is tight for μ-calculus.