The Tangled Derivative Logic of the Real Line and Zero-Dimensional Spaces

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Joint work with Ian Hodkinson

Prior paper:

- Spatial logic of modal mu-calculus and tangled closure operators.

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The tangle modality $\langle t \rangle$

Extend the basic modal language $\mathcal{L}_\Box$ to $\mathcal{L}_\Box^{\langle t \rangle}$ by allowing formation of the formula

$$\langle t \rangle \Gamma$$

when Γ is any finite non-empty set of formulas.

Semantics of $\langle t \rangle$ in a model on a Kripke frame (W, R):

$x \models \langle t \rangle \Gamma$ iff there is an endless R-path

$$x R x_1 \cdots x_n R x_{n+1} \cdots \cdots$$

in W with each member of Γ being true at x_n for infinitely many n.

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In a finite transitive frame:

an endless R-path eventually enters some non-degenerate cluster and stays there.

$x \models \langle t \rangle \Gamma$ iff x is R-related to some non-degenerate cluster C with each member of Γ true at some point of C.

$x \models \langle t \rangle \{\varphi\}$ iff there is a y with xRy and yRy and $y \models \varphi$

In a finite S4-model

$x \models \langle t \rangle \{\varphi\}$ iff there is a y with xRy and $y \models \varphi$
iff $x \models \Diamond\varphi$

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The modal mu-calculus language $\mathcal{L}^{\mu}_{\Box}$

Allows formation of the least fixed point formula

$$\mu p.\varphi$$

when p is positive in φ.

The greatest fixed point formula $\nu p.\varphi$ is

$$\neg\mu p.\varphi(\neg p/p).$$

Semantics in a model on a frame or space:

$[\![\mu p.\varphi]\!]$ is the least fixed point of the function $S \mapsto [\![\varphi]\!]_{p:=S}$

$$[\![\mu p.\varphi]\!] = \bigcap\{S \subseteq W : [\![\varphi]\!]_{p:=S} \subseteq S\}$$

$[\![\nu p.\varphi]\!]$ is the greatest fixed point:

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$\langle t \rangle \Gamma$ is definable in $\mathcal{L}^{\mu}_{\Box}$

In any model on a transitive frame,

$$[\![\langle t \rangle \Gamma]\!] = \bigcup \{ S \subseteq W : S \subseteq \bigcap_{\gamma \in \Gamma} R^{-1}([\![\gamma]\!] \cap S) \}$$

i.e. $[\![\langle t \rangle \Gamma]\!]$ is the largest set S such that

$$\text{for all } \gamma \in \Gamma, \quad S \subseteq R^{-1}([\![\gamma]\!] \cap S).$$

But $R^{-1}[\![\varphi]\!] = [\![\Diamond \varphi]\!]$, and $\bigcap$ interprets $\bigwedge$,
so $\langle t \rangle \Gamma$ has the same meaning as the $\mathcal{L}^{\mu}_{\Box}$-formula

$$\nu p. \bigwedge_{\gamma \in \Gamma} \Diamond(\gamma \wedge p)$$

Suggests a topological semantics: replace R^{-1} by closure

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Origin of the tangle modality:

van Benthem 1976

The bisimulation-invariant fragment of first-order logic is equivalent to $\mathcal{L}_{\Box}$.

This holds relative to any elementary class of frames (e.g. transitive).
And relative to the class of all finite frames [Rosen 1997]

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The bisimulation-invariant fragment of monadic second-order logic is equivalent to $\mathcal{L}^{\mu}_{\Box}$.

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over the class of finite transitive frames, the bisimulation-invariant fragment of monadic second-order logic collapses to that of first-order logic, with both fragments, and $\mathcal{L}^{\mu}_{\Box}$, being equivalent to the tangle extension $\mathcal{L}^{\langle t \rangle}_{\Box}$.

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Fernández-Duque 2011

- coined the name “tangle”.
- axiomatised the $\mathcal{L}_{\Box}^{\langle t \rangle}$-logic of the class of all (finite) S4-frames, as S4 +

$$\text{Fix: } \langle t \rangle \Gamma \to \Diamond(\gamma \wedge \langle t \rangle \Gamma), \quad \text{all } \gamma \in \Gamma.$$

$$\text{Ind: } \Box(\varphi \to \bigwedge_{\gamma \in \Gamma} \Diamond(\gamma \wedge \varphi)) \to (\varphi \to \langle t \rangle \Gamma).$$

- provided its topological interpretation, with closure in place of R^{-1}.

The derivative modality language $\mathcal{L}_{[d]}$

Replace $\Box$ and $\Diamond$ by $[d]$ and $\langle d \rangle$, with $[\![\langle d \rangle \varphi]\!] = R^{-1}[\![\varphi]\!]$

Define $\boxdot\varphi$ as $\varphi \wedge [d]\varphi$, and $\Diamond\varphi = \varphi \vee \langle d \rangle \varphi$.

In a topological space X, the derivative of a subset S is

$$\mathrm{deriv}\, S = \{x \in X : x \text{ is a limit point of } S\}.$$

$x \in \mathrm{deriv}\, S$ iff every neighbourhood O of x has $(O \setminus \{x\}) \cap S \neq \emptyset$.

In a model on X, $[\![\langle d \rangle \varphi]\!] = \mathrm{deriv}[\![\varphi]\!]$, so

$x \models \langle d \rangle \varphi$ iff every punctured neighbourhood of x intersects $[\![\varphi]\!]$,

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- $[\![\Box\varphi]\!]$ = the interior of $[\![\varphi]\!]$. $[\![\Diamond\varphi]\!]$ = the closure of $[\![\varphi]\!]$.
- Validity of the R-transitivity axiom

$$4: \quad \langle d\rangle\langle d\rangle\varphi \to \langle d\rangle\varphi$$

holds iff X is a T_D space, meaning $\mathrm{deriv}\{x\}$ is always closed. [*Aull & Thron* 1962]

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Shehtman 1990:
Derived sets in Euclidean spaces and modal logic.

Proved

- the $\mathcal{L}_{[d]}$-logic of every zero-dimensional separable dense-in-itself metric space is KD4.
- the $\mathcal{L}_{[d]}$-logic of the Euclidean space $\mathbb{R}^n$ for any $n \geq 2$ is

$$\text{KD4} \ + \ \text{G}_1 : \langle d\rangle p \wedge \langle d\rangle\neg p \to \langle d\rangle(\Diamond p \wedge \Diamond\neg p)$$

Conjectured

- the $\mathcal{L}_{[d]}$-logic of the real line $\mathbb{R}$ is KD4 + G_2, where G_n is

$$\bigwedge_{i\leq n} \langle d\rangle Q_i \to \langle d\rangle\big(\bigwedge_{i\leq n} \Diamond\neg Q_i\big), \qquad \text{with } Q_i = p_i \wedge \bigwedge_{i\neq j\leq n} \neg p_j.$$

[Proven later by Shehtman, and by Lucero-Bryan]

Asked

- Is KD4G_1 the largest logic of any dense-in-itself metric space?

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Whereas,

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Defining $\langle t \rangle$ from $\langle dt \rangle$

In a topological space X, $\langle t \rangle \Gamma$ is equivalent to

$$(\bigwedge \Gamma) \vee \langle d \rangle (\bigwedge \Gamma) \vee \langle dt \rangle \Gamma$$

if, and only if X is a T_D space.

Main results of our AiML 2016 paper:

Let Lt be the logic that extends a logic L by the tangle axioms

Fix: $\langle dt\rangle\Gamma \to \langle d\rangle(\gamma \wedge \langle dt\rangle\Gamma)$

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Adding the universal modality $\forall$

L.U is the extension of L that has the universal modality $\forall$ with semantics

$$w \models \forall\varphi \text{ iff for all } v \in W,\ v \models \varphi,$$

the S5 axioms and rules for $\forall$, and the axiom $\forall\varphi \to [d]\varphi$.

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- The $\mathcal{L}^{\langle dt \rangle}_{[d]\forall}$-logic of $\mathbb{R}$ is KD4G$_2t$.UC, where C is the axiom

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Strong completeness: ‘consistent sets are satisfiable’

Any countable KD4t-consistent set Γ of $\mathcal{L}_{[d]}^{\langle dt\rangle}$-formulas is satisfiable in any zero-dimensional dense-in-itself metric space.

Can fail for frame and spatial semantics for “large enough” Γ:

$$\{\Diamond p_i : i < \kappa\} \cup \{\neg\Diamond(p_i \wedge p_j) : i < j < \kappa\}$$

Not satisfiable in frame $\mathcal{F}$ if $\kappa > \mathrm{card}\,\mathcal{F}$.

Not satisfiable in space X if $\kappa > 2^{\mathrm{card}\,X}$.

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Strong completeness can fail for Kripke semantics for countable Γ:

$$\Sigma = \{\Diamond p_0\} \cup \{\Box(p_{2n} \to \Diamond(p_{2n+1} \wedge q)), \Box(p_{2n+1} \to \Diamond(p_{2n+2} \wedge \neg q)) : n < \omega\}$$

$\Sigma \cup \{\neg\langle t\rangle\{q, \neg q\}\}$ is finitely satisfiable, so is K4t-consistent, but is not satisfiable in any Kripke model.

Also shows that in the canonical model for K4t, the ‘Truth Lemma’ fails.

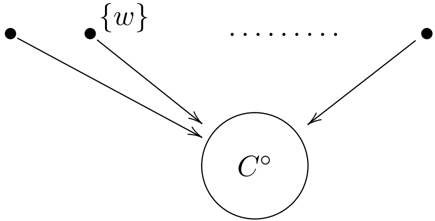
Proving a logic L is complete over space X:

1. Prove the finite model property for L over Kripke frames:
   if L $\nvdash \varphi$, then φ is falsifiable in some suitable finite frame $\mathcal{F} \models$L.

2. Construct a surjective d-morphism $f : X \twoheadrightarrow \mathcal{F}$:

$$f^{-1}(R^{-1}(S)) = \mathrm{deriv}\, f^{-1}(S).$$

Such an f preserves validity of formulas from X to $\mathcal{F}$, so $X \not\models \varphi$.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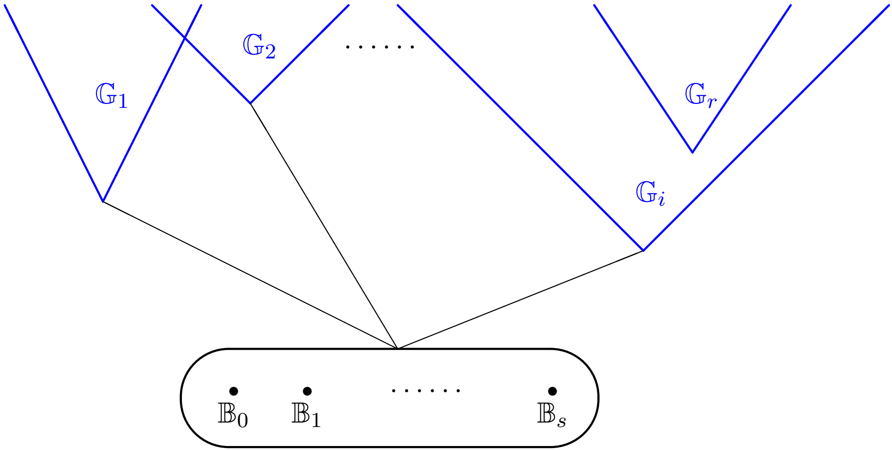
C°

Enncoding a d-morphism $X \twoheadrightarrow \mathcal{F}$,
when $\mathcal{F}$ is a point-generated S4-frame.



Modified Tarski Dissection Theorem

Let X be a dense-in-itself metric space.
Then X is dissectable:

Let $\mathbb{G}$ be a non-empty open subset of X, and let $r, s < \omega$.

Then $\mathbb{G}$ can be partitioned into non-empty subsets

$$\mathbb{G}_1, \ldots, \mathbb{G}_r, \mathbb{B}_0, \ldots, \mathbb{B}_s$$

such that the $\mathbb{G}_i$'s are all open and

$$\mathrm{cl}(\mathbb{G}_i) \setminus \mathbb{G}_i = \mathrm{deriv}(\mathbb{B}_j) = \mathrm{cl}(\mathbb{G}) \setminus (\mathbb{G}_1 \cup \cdots \cup \mathbb{G}_r).$$

Further dissections of a dense-in-itself metric X

1. Let $\mathbb{G}$ be a non-empty open subset of X, and let $k < \omega$. Then there are pairwise disjoint non-empty subsets $\mathbb{I}_0, \ldots, \mathbb{I}_k \subseteq \mathbb{G}$ satisfying

$$\operatorname{deriv} \mathbb{I}_i = \operatorname{cl}(\mathbb{G}) \setminus \mathbb{G} \quad \text{for each } i \leq k.$$

2. Let X be zero-dimensional.
If $\mathbb{G}$ is a non-empty open subset of X, and $n < \omega$, then $\mathbb{G}$ can be partitioned into non-empty open subsets $\mathbb{G}_0, \ldots, \mathbb{G}_n$ such that

$$\operatorname{cl}(\mathbb{G}_i) \setminus \mathbb{G}_i = \operatorname{cl}(\mathbb{G}) \setminus \mathbb{G} \text{ for each } i \leq n.$$