Embedding Formalisms: Hypersequents & Two-Level Systems of Rules

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logics Logical Methods in Computer Science

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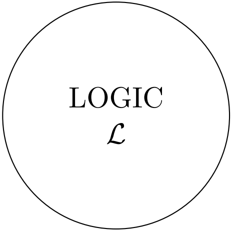
Logics, proof systems and formalisms



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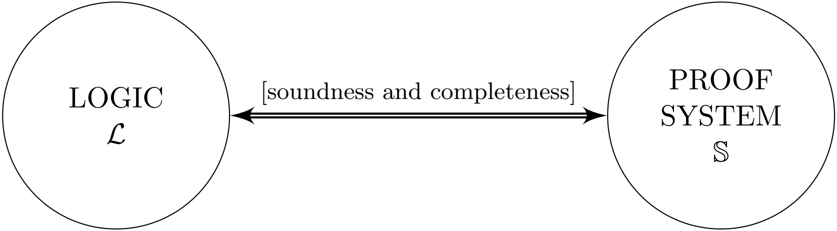

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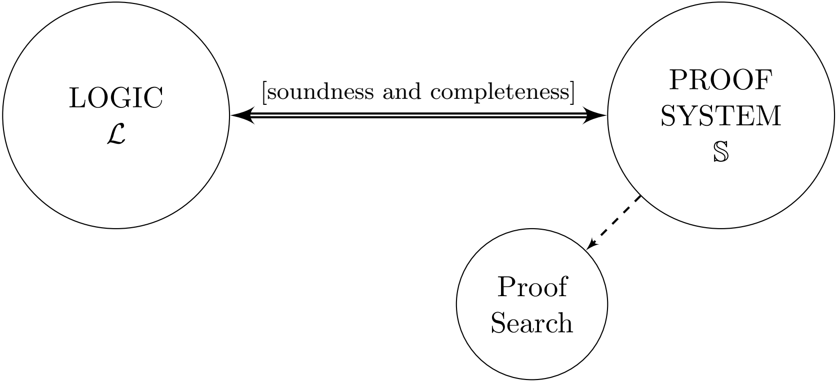

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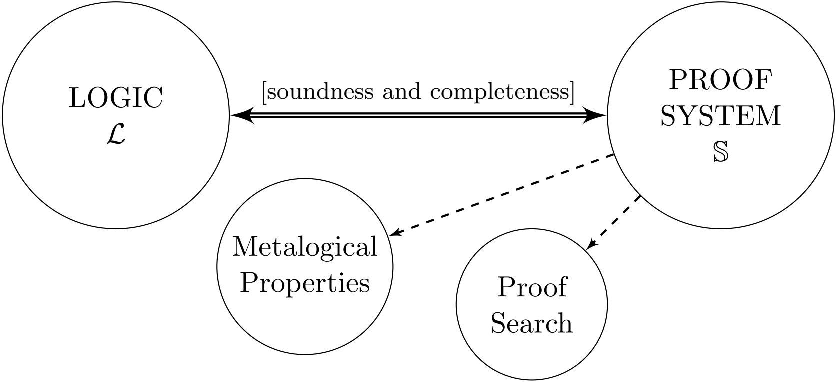

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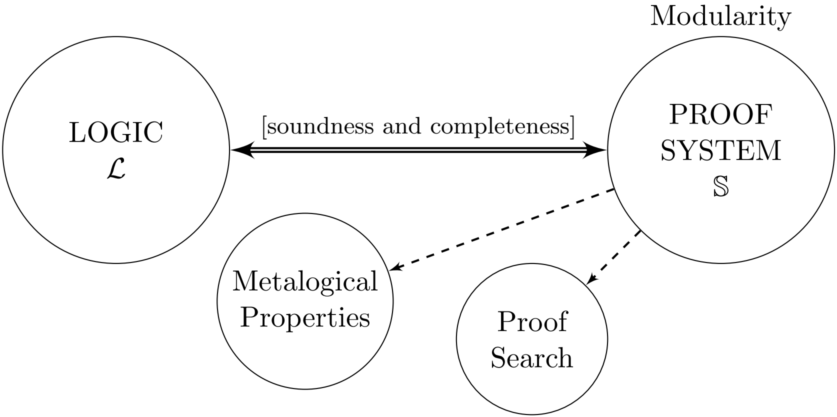

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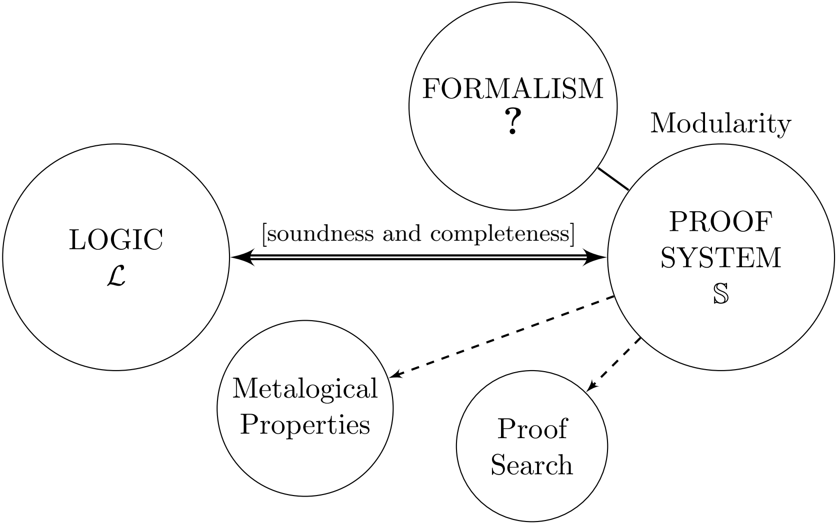

Good proof systems

ANALYTICITY

Derivations of a formula A only contain formulae which are subformulae of A

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MODULARITY

For a logic $\mathcal{L} + axioms$ we need:

1. base system $\mathbb{S}$ for $\mathcal{L}$
2. some (hopefully) analytic rules for the *axioms*

Adding one axiom to $\mathcal{L} \Rightarrow$ Adding some rules to $\mathbb{S}$

A jungle of formalisms

Many formalisms to capture logics

(sequents, hypersequents, labelled sequents,
nested sequents, display calculus, calculus of structures…)

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Embeddings between formalisms

$\Downarrow$

- Expressiveness relations
- Transfer of results
  (avoiding repetitions and possible mistakes)

[Wansing, 1998], [Fitting, 2012],
[Goré and Ramanayake, 2012], [Ramanayake 2015, 2016]. . .

Our context

To obtain **analytic** and **modular** (sequent-like) proof systems:

Extend the base proof system
by **analyticity preserving structural rules**
—those not referring to connectives—

$\Downarrow$

Modularity (fixed logical rules)

General translation methods from axioms to rules
[http://www.logic.at/people/lara/axiomcalc.html]

Sequents and beyond

Sequents are **simple** and **versatile**

$$\Gamma \Rightarrow \Delta$$

Sequents and beyond

Sequents are **simple** and **versatile**

$$\Gamma \Rightarrow \Delta$$

but **not enough** to define modular analytic proof systems **for many interesting logics**

Consider the axioms for **intermediate logics**:

$$\begin{cases} \neg\neg A \lor \neg A & \textit{Jankov} \\ (A \supset B) \lor (B \supset A) & \textit{Gödel} \\ A \lor (A \supset (B \lor (B \supset C))) & Bd_2 \\ \vdots & \end{cases}$$

No sequent structural rule can capture these axioms [Ciabattoni et al., 2012, Ann. Pure Appl. Logic]

So what? More structure!

The ***linearity axiom*** characterises *Gödel logic*

$$(A \supset B) \vee (B \supset A)$$

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The ***linearity axiom*** characterises *Gödel logic*

$$(A \supset B) \vee (B \supset A)$$

We can define structural rules based on the syntax of this axiom using two (simple) generalisations of sequents:

HYPERSEQUENTS
[Mints, 1968]
[Pottinger, 1983]
[Avron, 1987]

and

SYSTEMS OF RULES
[Negri, 2014]

Hypersequents [Mints, 1968], [Pottinger, 1983], [Avron, 1987]

$$\frac{\text{Multisets of sequents (interpreted disjunctively)}}{\Gamma_1 \Rightarrow \Delta_1 \mid \ldots \mid \Gamma_n \Rightarrow \Delta_n}$$

Hypersequents [Mints, 1968], [Pottinger, 1983], [Avron, 1987]

Multisets of sequents (interpreted disjunctively)

$$\Gamma_1 \Rightarrow \Delta_1 \mid \ldots \mid \Gamma_n \Rightarrow \Delta_n$$

We can represent the *linearity axiom* as

$$(A \supset B) \vee (B \supset A)$$

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We can represent the *linearity axiom* as

$$A \Rightarrow B \mid B \Rightarrow A$$

and transform this into the rule

$$\frac{\mathcal{G} \mid B, \Gamma_1 \Rightarrow \Delta_1 \qquad \mathcal{G} \mid A, \Gamma_2 \Rightarrow \Delta_2}{\mathcal{G} \mid A, \Gamma_1 \Rightarrow \Delta_1 \mid B, \Gamma_2 \Rightarrow \Delta_2}$$

Example of hypersequent derivation

$$\frac{\mathcal{G} \mid B, \Gamma_1 \Rightarrow \Delta_1 \qquad \mathcal{G} \mid A, \Gamma_2 \Rightarrow \Delta_2}{\mathcal{G} \mid A, \Gamma_1 \Rightarrow \Delta_1 \mid B, \Gamma_2 \Rightarrow \Delta_2}\ (com) \qquad \frac{\mathcal{G} \mid \Gamma \Rightarrow \Delta \mid \Gamma \Rightarrow \Delta}{\mathcal{G} \mid \Gamma \Rightarrow \Delta}\ (EC)$$

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\overline{B \Rightarrow B}\ init. \qquad \overline{A \Rightarrow A}\ init.}{A \Rightarrow B \mid B \Rightarrow A}\ (com)}{A \Rightarrow B \mid \Rightarrow B \supset A}\ (\supset r)}{\Rightarrow A \supset B \mid \Rightarrow B \supset A}\ (\supset r)}{\Rightarrow A \supset B \mid \Rightarrow (A \supset B) \vee (B \supset A)}\ (\vee r)}{\Rightarrow (A \supset B) \vee (B \supset A) \mid \Rightarrow (A \supset B) \vee (B \supset A)}\ (\vee r)}{\Rightarrow (A \supset B) \vee (B \supset A)}\ (EC)$$

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Systems of rules [Negri, 2014, J. Logic Comput]

Sets of rules related by applicability constraints

Very expressive formalism

Systems of rules on labelled sequents (e.g. "$xRy, \Gamma \Rightarrow \Delta, y : A$") capture all normal modal logics formalised by Sahlqvist formulae

Systems of rules

Sets of rules related by applicability constraints

We will only consider
purely syntactical and **two-level systems**:

$$\dfrac{\Gamma_1^1 \Rightarrow \Delta_1^1 \ \ldots \ \Gamma_1^{n_1} \Rightarrow \Delta_1^{n_1}}{\Gamma_1 \Rightarrow \Delta_1}\ (top_1) \qquad \dfrac{\Gamma_k^1 \Rightarrow \Delta_k^1 \ \ldots \ \Gamma_k^{n_k} \Rightarrow \Delta_k^{n_k}}{\Gamma_k \Rightarrow \Delta_k}\ (top_k)$$

$$\dfrac{\begin{matrix}\Gamma_1 \Rightarrow \Delta_1 \\ \vdots \\ \Gamma \Rightarrow \Delta\end{matrix} \qquad \ldots \qquad \begin{matrix}\Gamma_k \Rightarrow \Delta_k \\ \vdots \\ \Gamma \Rightarrow \Delta\end{matrix}}{\Gamma \Rightarrow \Delta}\ (bottom)$$

Example of system

We represent the axiom $(A \supset B) \vee (B \supset A)$ as the system

$$\dfrac{\dfrac{\dfrac{B, \Gamma_1 \Rightarrow \Delta_1}{A, \Gamma_1 \Rightarrow \Delta_1}\ (com_1)}{\vdots}\quad \Gamma \Rightarrow \Delta \qquad \dfrac{\dfrac{A, \Gamma_2 \Rightarrow \Delta_2}{B, \Gamma_2 \Rightarrow \Delta_2}\ (com_2)}{\vdots}\quad \Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta}\ (end)$$

shown to be analytic for atomic A and B
[Negri, 2014, J. Logic Comput.]

Example of systems of rules derivation

$$\dfrac{\dfrac{\dfrac{B, \Gamma_1 \Rightarrow \Delta_1}{A, \Gamma_1 \Rightarrow \Delta_1}\,(com_1)}{\vdots}\quad \Gamma \Rightarrow \Delta \qquad\qquad \dfrac{\dfrac{A, \Gamma_2 \Rightarrow \Delta_2}{B, \Gamma_2 \Rightarrow \Delta_2}\,(com_1)}{\vdots}\quad \Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta}\,(end)$$

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{B \Rightarrow B}\,init.}{A \Rightarrow B}\,(com_1)}{\Rightarrow A \supset B}\,(\supset r)}{\Rightarrow (A \supset B) \vee (B \supset A)}\,(\vee r) \qquad \dfrac{\dfrac{\dfrac{\dfrac{}{A \Rightarrow A}\,init.}{B \Rightarrow A}\,(com_2)}{\Rightarrow B \supset A}\,(\supset r)}{\Rightarrow (A \supset B) \vee (B \supset A)}\,(\vee r)}{\Rightarrow (A \supset B) \vee (B \supset A)}\,(end)$$

Example of systems of rules derivation

$$\dfrac{\begin{array}{c}\dfrac{B,\Gamma_1 \Rightarrow \Delta_1}{A,\Gamma_1 \Rightarrow \Delta_1}\,(com_1)\\ \vdots \\ {\color{red}\Gamma \Rightarrow \Delta}\end{array} \qquad \begin{array}{c}\dfrac{A,\Gamma_2 \Rightarrow \Delta_2}{B,\Gamma_2 \Rightarrow \Delta_2}\,(com_1)\\ \vdots \\ {\color{red}\Gamma \Rightarrow \Delta}\end{array}}{\Gamma \Rightarrow \Delta}\,(end)$$

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Example of systems of rules derivation

$$
\dfrac{
\begin{array}{c}
\dfrac{B, \Gamma_1 \Rightarrow \Delta_1}{A, \Gamma_1 \Rightarrow \Delta_1}\ (com_1) \\
\vdots \\
\Gamma \Rightarrow \Delta
\end{array}
\qquad\qquad
\begin{array}{c}
\dfrac{A, \Gamma_2 \Rightarrow \Delta_2}{B, \Gamma_2 \Rightarrow \Delta_2}\ (com_1) \\
\vdots \\
\Gamma \Rightarrow \Delta
\end{array}
}{\Gamma \Rightarrow \Delta}\ (end)
$$

$$
\dfrac{
\dfrac{\dfrac{\dfrac{\dfrac{}{B \Rightarrow B}\ init.}{A \Rightarrow B}\ (com_1)}{\Rightarrow A \supset B}\ (\supset r)}{\Rightarrow (A \supset B) \vee (B \supset A)}\ (\vee r)
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}{\Rightarrow (A \supset B) \vee (B \supset A)}\ (end)
$$

Example of systems of rules derivation

$$\dfrac{\dfrac{\dfrac{B, \Gamma_1 \Rightarrow \Delta_1}{A, \Gamma_1 \Rightarrow \Delta_1}\ (com_1)}{\vdots}\quad \dfrac{\dfrac{A, \Gamma_2 \Rightarrow \Delta_2}{B, \Gamma_2 \Rightarrow \Delta_2}\ (com_1)}{\vdots}}{}$$

$$\dfrac{\Gamma \Rightarrow \Delta \qquad \Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta}\ (end)$$

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$$
\dfrac{
\begin{array}{c}
\dfrac{{\color{red}B}, \Gamma_1 \Rightarrow \Delta_1}{A, \Gamma_1 \Rightarrow \Delta_1}\,(com_1) \\
\vdots \\
\Gamma \Rightarrow \Delta
\end{array}
\qquad\qquad
\begin{array}{c}
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\vdots \\
\Gamma \Rightarrow \Delta
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$$

$$
\dfrac{
\dfrac{
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\dfrac{\dfrac{}{{\color{red}B} \Rightarrow B}\,init.}{A \Rightarrow B}\,(com_1)
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\qquad
\dfrac{
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\dfrac{\dfrac{}{A \Rightarrow A}\,init.}{{\color{red}B} \Rightarrow A}\,(com_2)
}{\Rightarrow B \supset A}\,(\supset r)
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Example of systems of rules derivation

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A casual resemblance?

$$\frac{\mathcal{G} \mid B, \Gamma_1 \Rightarrow \Delta_1 \qquad \mathcal{G} \mid A, \Gamma_2 \Rightarrow \Delta_2}{\mathcal{G} \mid A, \Gamma_1 \Rightarrow \Delta_1 \mid B, \Gamma_2 \Rightarrow \Delta_2}$$

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The two formalisms seem to be related

A possible connection is suggested in [Negri, 2014, J. Logic Comput.]

A casual resemblance?

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Can we fully formalise this intuition?

Where does this lead to?

Our Embedding

Rules translation

- Any hypersequent rule can be rewritten as a two-level system of rules
- Any two-level system of rules can be rewritten as a hypersequent rule

$$\frac{\mathcal{G} \mid \Gamma'_1 \Rightarrow \Delta'_1 \qquad \dots \qquad \mathcal{G} \mid \Gamma'_k \Rightarrow \Delta'_k}{\mathcal{G} \mid \Gamma_1 \Rightarrow \Delta_1 \mid \dots \mid \Gamma_n \Rightarrow \Delta_n}$$

$\mathcal{S}_1, \dots, \mathcal{S}_n$ sets of sequents $\Downarrow$ $\mathcal{S}_1 \cup \dots \cup \mathcal{S}_n = \{\Gamma'_i \Rightarrow \Delta'_i\}_{1 \leq i \leq k}$

$$\frac{\begin{array}{c}\dfrac{\mathcal{S}_1}{\Gamma_1 \Rightarrow \Delta_1} \\ \vdots \\ \Gamma \Rightarrow \Delta\end{array} \qquad \cdots \qquad \begin{array}{c}\dfrac{\mathcal{S}_n}{\Gamma_n \Rightarrow \Delta_n} \\ \vdots \\ \Gamma \Rightarrow \Delta\end{array}}{\Gamma \Rightarrow \Delta}$$

Rules translation

- Any hypersequent rule can be rewritten as a two-level system of rules
- Any two-level system of rules can be rewritten as a hypersequent rule

$$\frac{\mathcal{G} \mid \Gamma_1' \Rightarrow \Delta_1' \qquad \dots \qquad \mathcal{G} \mid \Gamma_k' \Rightarrow \Delta_k'}{\mathcal{G} \mid \Gamma_1 \Rightarrow \Delta_1 \mid \dots \mid \Gamma_n \Rightarrow \Delta_n}$$

$\mathcal{S}_1, \dots, \mathcal{S}_n$ sets of sequents $\Uparrow$ $\mathcal{S}_1 \cup \dots \cup \mathcal{S}_n = \{\Gamma_i' \Rightarrow \Delta_i'\}_{1 \leq i \leq k}$

$$\dfrac{\dfrac{\dfrac{\mathcal{S}_1}{\Gamma_1 \Rightarrow \Delta_1}}{\vdots}}{\Gamma \Rightarrow \Delta} \quad \cdots \quad \dfrac{\dfrac{\dfrac{\mathcal{S}_n}{\Gamma_n \Rightarrow \Delta_n}}{\vdots}}{\Gamma \Rightarrow \Delta}$$

$$\frac{\Gamma \Rightarrow \Delta \qquad \cdots \qquad \Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta}$$

Derivations translation

- Any hypersequent derivation can be translated into a two-level systems of rules derivation
- Any two-level systems of rules derivation can be translated into a hypersequent derivation

The steps of the translation

Hypersequents $\xrightarrow{\textit{translation}}$ **Systems of Rules**

A. Pre-processing
   1. Shift down (EC) to the root
   2. Shift down (EW) as much as possible

B. Bottom-up translation rule by rule
   of the (EC)-free part of the derivation

C. Application of the bottom rules to simulate (EC)

The steps of the translation II

$$\textbf{Hypersequents} \overset{\textit{translation}}{\Longleftarrow\!=\!=\!=} \textbf{Systems of Rules}$$

Top-down translation rule by rule (+ (EW) for contexts)

Important lemmata

- We can translate systems of rules one by one (no entanglement of systems)
- The branching of hypersequent components and the branching of systems of rules match

Example

$$
\dfrac{
  \dfrac{A \Rightarrow A}{A \Rightarrow A \mid B \Rightarrow A \wedge B}\,(EW)
  \qquad
  \dfrac{
    \dfrac{B \Rightarrow B \quad A \Rightarrow A}{A \Rightarrow B \mid B \Rightarrow A}\,(com)
    \quad
    \dfrac{B \Rightarrow B}{A \Rightarrow B \mid B \Rightarrow B}\,(EW)
  }{A \Rightarrow B \mid B \Rightarrow A \wedge B}\,(\wedge r)
}{
  \dfrac{
    \dfrac{
      \dfrac{
        \dfrac{A \Rightarrow A \wedge B \mid B \Rightarrow A \wedge B}{A \Rightarrow A \wedge B \mid\Rightarrow B \supset (A \wedge B)}\,(\supset r)
      }{\Rightarrow A \supset (A \wedge B) \mid \;\Rightarrow B \supset (A \wedge B)}\,(\supset r)
    }{\Rightarrow A \supset (A \wedge B) \mid\Rightarrow (A \supset (A \wedge B)) \vee (B \supset (A \wedge B))}\,(\vee r)
  }{
    \dfrac{\Rightarrow (A \supset (A \wedge B)) \vee (B \supset (A \wedge B)) \mid \;\Rightarrow (A \supset (A \wedge B)) \vee (B \supset (A \wedge B))}{\Rightarrow (A \supset (A \wedge B)) \vee (B \supset (A \wedge B))}\,(EC)
  }\,(\vee r)
}\,(\wedge r)
$$

$$\Updownarrow$$

$$
\dfrac{
  \dfrac{
    \dfrac{
      \dfrac{A \Rightarrow A \quad \dfrac{B \Rightarrow B}{A \Rightarrow B}\,(com_1)}{A \Rightarrow A \wedge B}\,(\wedge r)
    }{\Rightarrow A \supset (A \wedge B)}\,(\supset r)
  }{\Rightarrow (A \supset (A \wedge B)) \vee (B \supset (A \wedge B))}\,(\vee r)
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      \dfrac{\dfrac{A \Rightarrow A}{B \Rightarrow A}\,(com_2) \quad B \Rightarrow B}{B \Rightarrow A \wedge B}\,(\wedge r)
    }{\Rightarrow B \supset (A \wedge B)}\,(\supset r)
  }{\Rightarrow (A \supset (A \wedge B)) \vee (B \supset (A \wedge B))}\,(\vee r)
}{\Rightarrow (A \supset (A \wedge B)) \vee (B \supset (A \wedge B))}\,(end)
$$

Example

$$
\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{A \Rightarrow A}{A \Rightarrow A \mid B \Rightarrow A \wedge B}\,(EW) \qquad \dfrac{\dfrac{B \Rightarrow B \quad A \Rightarrow A}{A \Rightarrow B \mid B \Rightarrow A}\,(com) \qquad \dfrac{B \Rightarrow B}{A \Rightarrow B \mid B \Rightarrow B}\,(EW)}{A \Rightarrow B \mid B \Rightarrow A \wedge B}\,(\wedge r)}{A \Rightarrow A \wedge B \mid B \Rightarrow A \wedge B}\,(\wedge r)}{A \Rightarrow A \wedge B \mid \Rightarrow B \supset (A \wedge B)}\,(\supset r)}{\Rightarrow A \supset (A \wedge B) \mid \Rightarrow B \supset (A \wedge B)}\,(\supset r)}{\Rightarrow A \supset (A \wedge B) \mid \Rightarrow (A \supset (A \wedge B)) \vee (B \supset (A \wedge B))}\,(\vee r)}{\color{red}{\Rightarrow (A \supset (A \wedge B)) \vee (B \supset (A \wedge B))} \mid \color{blue}{\Rightarrow (A \supset (A \wedge B)) \vee (B \supset (A \wedge B))}}\,(\vee r)}{\Rightarrow (A \supset (A \wedge B)) \vee (B \supset (A \wedge B))}\,(EC)
$$

$$
\color{red}{\Updownarrow}
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$$
\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{A \Rightarrow A \qquad \dfrac{B \Rightarrow B}{A \Rightarrow B}\,(com_1)}{A \Rightarrow A \wedge B}\,(\wedge r)}{\Rightarrow A \supset (A \wedge B)}\,(\supset r)}{\color{red}{\Rightarrow (A \supset (A \wedge B)) \vee (B \supset (A \wedge B))}}\,(\vee r) \qquad \dfrac{\dfrac{\dfrac{\dfrac{A \Rightarrow A}{B \Rightarrow A}\,(com_2) \qquad B \Rightarrow B}{B \Rightarrow A \wedge B}\,(\wedge r)}{\Rightarrow B \supset (A \wedge B)}\,(\supset r)}{\color{blue}{\Rightarrow (A \supset (A \wedge B)) \vee (B \supset (A \wedge B))}}\,(\vee r)}{\Rightarrow (A \supset (A \wedge B)) \vee (B \supset (A \wedge B))}\,(end)}
$$

Example

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\dfrac{A \Rightarrow A}{A \Rightarrow A \mid B \Rightarrow A \wedge B}\,(EW)
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\dfrac{
\dfrac{B \Rightarrow B \quad A \Rightarrow A}{A \Rightarrow B \mid B \Rightarrow A}\,(com)
\qquad
\dfrac{B \Rightarrow B}{A \Rightarrow B \mid B \Rightarrow B}\,(EW)
}{A \Rightarrow B \mid B \Rightarrow A \wedge B}\,(\wedge r)
}{A \Rightarrow A \wedge B \mid B \Rightarrow A \wedge B}\,(\wedge r)
}{A \Rightarrow A \wedge B \mid \Rightarrow B \supset (A \wedge B)}\,(\supset r)
}{\Rightarrow A \supset (A \wedge B) \mid \Rightarrow B \supset (A \wedge B)}\,(\supset r)
}{\Rightarrow A \supset (A \wedge B) \mid \Rightarrow (A \supset (A \wedge B)) \vee (B \supset (A \wedge B))}\,(\vee r)
}{\Rightarrow (A \supset (A \wedge B)) \vee (B \supset (A \wedge B)) \mid \Rightarrow (A \supset (A \wedge B)) \vee (B \supset (A \wedge B))}\,(\vee r)
}{\Rightarrow (A \supset (A \wedge B)) \vee (B \supset (A \wedge B))}\,(EC)
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$$\Updownarrow$$

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\dfrac{A \Rightarrow A \qquad \dfrac{B \Rightarrow B}{A \Rightarrow B}\,(com_1)}{A \Rightarrow A \wedge B}\,(\wedge r)
}{\Rightarrow A \supset (A \wedge B)}\,(\supset r)
}{\Rightarrow (A \supset (A \wedge B)) \vee (B \supset (A \wedge B))}\,(\vee r)
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\dfrac{\dfrac{A \Rightarrow A}{B \Rightarrow A}\,(com_2) \qquad B \Rightarrow B}{B \Rightarrow A \wedge B}\,(\wedge r)
}{\Rightarrow B \supset (A \wedge B)}\,(\supset r)
}{\Rightarrow (A \supset (A \wedge B)) \vee (B \supset (A \wedge B))}\,(\vee r)
}{\Rightarrow (A \supset (A \wedge B)) \vee (B \supset (A \wedge B))}\,(end)
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Example

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          \dfrac{A \Rightarrow A}{A \Rightarrow A \mid B \Rightarrow A \wedge B}\;(EW)
          \qquad
          \dfrac{
            \dfrac{B \Rightarrow B \quad A \Rightarrow A}{A \Rightarrow B \mid B \Rightarrow A}\;(com)
            \qquad
            \dfrac{B \Rightarrow B}{A \Rightarrow B \mid B \Rightarrow B}\;(EW)
          }{A \Rightarrow B \mid B \Rightarrow A \wedge B}\;(\wedge r)
        }{A \Rightarrow A \wedge B \mid {\color{blue} B \Rightarrow A \wedge B}}\;(\wedge r)
      }{{\color{red} A \Rightarrow A \wedge B} \mid \Rightarrow B \supset (A \wedge B)}\;(\supset r)
    }{\Rightarrow A \supset (A \wedge B) \mid \;\Rightarrow B \supset (A \wedge B)}\;(\supset r)
  }{\Rightarrow A \supset (A \wedge B) \mid \Rightarrow (A \supset (A \wedge B)) \vee (B \supset (A \wedge B))}\;(\vee r)
}{
  \dfrac{\Rightarrow (A \supset (A \wedge B)) \vee (B \supset (A \wedge B)) \mid \;\Rightarrow (A \supset (A \wedge B)) \vee (B \supset (A \wedge B))}{\Rightarrow (A \supset (A \wedge B)) \vee (B \supset (A \wedge B))}\;(EC)
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{\color{red} \Updownarrow}
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      \dfrac{A \Rightarrow A \qquad \dfrac{B \Rightarrow B}{A \Rightarrow B}\;(com_1)}{{\color{red} A \Rightarrow A \wedge B}}\;(\wedge r)
    }{\Rightarrow A \supset (A \wedge B)}\;(\supset r)
  }{\Rightarrow (A \supset (A \wedge B)) \vee (B \supset (A \wedge B))}\;(\vee r)
  \qquad
  \dfrac{
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      \dfrac{\dfrac{A \Rightarrow A}{B \Rightarrow A}\;(com_2) \qquad B \Rightarrow B}{{\color{blue} B \Rightarrow A \wedge B}}\;(\wedge r)
    }{\Rightarrow B \supset (A \wedge B)}\;(\supset r)
  }{\Rightarrow (A \supset (A \wedge B)) \vee (B \supset (A \wedge B))}\;(\vee r)
}{\Rightarrow (A \supset (A \wedge B)) \vee (B \supset (A \wedge B))}\;(end)
$$

Example

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\dfrac{A \Rightarrow A}{A \Rightarrow A \mid B \Rightarrow A \wedge B}\,(EW)
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\dfrac{
\dfrac{B \Rightarrow B \quad A \Rightarrow A}{A \Rightarrow B \mid B \Rightarrow A}\,(com)
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\dfrac{B \Rightarrow B}{A \Rightarrow B \mid B \Rightarrow B}\,(EW)
}{A \Rightarrow B \mid B \Rightarrow A \wedge B}\,(\wedge r)
}{A \Rightarrow A \wedge B \mid B \Rightarrow A \wedge B}\,(\wedge r)
}{A \Rightarrow A \wedge B \mid \Rightarrow B \supset (A \wedge B)}\,(\supset r)
}{\Rightarrow A \supset (A \wedge B) \mid \Rightarrow B \supset (A \wedge B)}\,(\supset r)
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}{\Rightarrow (A \supset (A \wedge B)) \vee (B \supset (A \wedge B)) \mid \Rightarrow (A \supset (A \wedge B)) \vee (B \supset (A \wedge B))}\,(\vee r)
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$$\Updownarrow$$

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\dfrac{A \Rightarrow A \quad \dfrac{B \Rightarrow B}{A \Rightarrow B}\,(com_1)}{A \Rightarrow A \wedge B}\,(\wedge r)
}{\Rightarrow A \supset (A \wedge B)}\,(\supset r)
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\dfrac{\dfrac{A \Rightarrow A}{B \Rightarrow A}\,(com_2) \quad B \Rightarrow B}{B \Rightarrow A \wedge B}\,(\wedge r)
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$$

Example

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\dfrac{
  \dfrac{A \Rightarrow A}{A \Rightarrow A \mid B \Rightarrow A \wedge B}\,(EW)
  \qquad
  \dfrac{
    \dfrac{B \Rightarrow B \quad A \Rightarrow A}{A \Rightarrow B \mid B \Rightarrow A}\,(com)
    \quad
    \dfrac{B \Rightarrow B}{A \Rightarrow B \mid B \Rightarrow B}\,(EW)
  }{A \Rightarrow B \mid B \Rightarrow A \wedge B}\,(\wedge r)
}{
  \dfrac{
    \dfrac{
      \dfrac{
        \dfrac{A \Rightarrow A \wedge B \mid B \Rightarrow A \wedge B}{A \Rightarrow A \wedge B \mid \Rightarrow B \supset (A \wedge B)}\,(\supset r)
      }{\Rightarrow A \supset (A \wedge B) \mid \Rightarrow B \supset (A \wedge B)}\,(\supset r)
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  }{
    \dfrac{\Rightarrow (A \supset (A \wedge B)) \vee (B \supset (A \wedge B)) \mid \Rightarrow (A \supset (A \wedge B)) \vee (B \supset (A \wedge B))}{\Rightarrow (A \supset (A \wedge B)) \vee (B \supset (A \wedge B))}\,(EC)
  }\,(\vee r)
}\,(\wedge r)
$$

$$\Updownarrow$$

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      \dfrac{A \Rightarrow A \quad \dfrac{B \Rightarrow B}{A \Rightarrow B}\,(com_1)}{A \Rightarrow A \wedge B}\,(\wedge r)
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\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{A \Rightarrow A}{A \Rightarrow A \mid B \Rightarrow A \wedge B}\,(EW) \qquad \dfrac{\dfrac{B \Rightarrow B \quad A \Rightarrow A}{A \Rightarrow B \mid B \Rightarrow A}\,(com) \qquad \dfrac{B \Rightarrow B}{A \Rightarrow B \mid B \Rightarrow B}\,(EW)}{A \Rightarrow B \mid B \Rightarrow A \wedge B}\,(\wedge r)}{A \Rightarrow A \wedge B \mid B \Rightarrow A \wedge B}\,(\wedge r)}{A \Rightarrow A \wedge B \mid \Rightarrow B \supset (A \wedge B)}\,(\supset r)}{\Rightarrow A \supset (A \wedge B) \mid \Rightarrow B \supset (A \wedge B)}\,(\supset r)}{\Rightarrow A \supset (A \wedge B) \mid \Rightarrow (A \supset (A \wedge B)) \vee (B \supset (A \wedge B))}\,(\vee r)}{\Rightarrow (A \supset (A \wedge B)) \vee (B \supset (A \wedge B)) \mid \Rightarrow (A \supset (A \wedge B)) \vee (B \supset (A \wedge B))}\,(\vee r)}{\Rightarrow (A \supset (A \wedge B)) \vee (B \supset (A \wedge B))}\,(EC)
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$$
\dfrac{\dfrac{\dfrac{\dfrac{A \Rightarrow A \quad \dfrac{B \Rightarrow B}{A \Rightarrow B}\,(com_1)}{A \Rightarrow A \wedge B}\,(\wedge r)}{\Rightarrow A \supset (A \wedge B)}\,(\supset r)}{\Rightarrow (A \supset (A \wedge B)) \vee (B \supset (A \wedge B))}\,(\vee r) \qquad \dfrac{\dfrac{\dfrac{\dfrac{A \Rightarrow A}{B \Rightarrow A}\,(com_2) \quad B \Rightarrow B}{B \Rightarrow A \wedge B}\,(\wedge r)}{\Rightarrow B \supset (A \wedge B)}\,(\supset r)}{\Rightarrow (A \supset (A \wedge B)) \vee (B \supset (A \wedge B))}\,(\vee r)}{\Rightarrow (A \supset (A \wedge B)) \vee (B \supset (A \wedge B))}\,(end)
$$

How far does this go?

The two formalisms are equivalent
w.r.t. intermediate logics, but…

The embedding only requires (EW), (EC), (EE)
and premisses with at most one active component

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w.r.t. intermediate logics, but…

The embedding only requires (EW), (EC), (EE)
and premisses with at most one active component

$\Downarrow$

can be naturally extended to other calculi
as those in [Kurokawa, 2013][Lahav, 2013][Indrzejczak, 2015]
e.g. for the logics S5, S4.2, S4.3, K4.2 and K4.3

Applications of the Embedding

Systems of rules made local

By the embedding we can
represent two-level systems of rules locally:

$$\dfrac{\dfrac{\dfrac{B, \Gamma_1 \Rightarrow \Delta_1}{A, \Gamma_1 \Rightarrow \Delta_1}}{\vdots \atop \Gamma \Rightarrow \Delta} \qquad \dfrac{\dfrac{A, \Gamma_2 \Rightarrow \Delta_2}{B, \Gamma_2 \Rightarrow \Delta_2}}{\vdots \atop \Gamma \Rightarrow \Delta}}{\Gamma \Rightarrow \Delta} \quad \rightsquigarrow \quad \dfrac{\mathcal{G} \mid B, \Gamma_1 \Rightarrow \Delta_1 \qquad \mathcal{G} \mid A, \Gamma_2 \Rightarrow \Delta_2}{\mathcal{G} \mid A, \Gamma_1 \Rightarrow \Delta_1 \mid B, \Gamma_2 \Rightarrow \Delta_2}$$

Systems of rules made analytic

In [Negri, 2014, J. Logic Comput.] it is proved that systems of rules acting on atoms preserve cut-elimination

Given **any** two-level system of rules we can:

1. Translate the system of rules into a hypersequent rule [embedding]

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1. Translate the system of rules into a hypersequent rule [embedding]
2. Apply the “completion” procedure [Ciabattoni et al., 2008, LICS]
3. Translate the result back [embedding]

We thus obtain an **analytic 2-level system of rules**

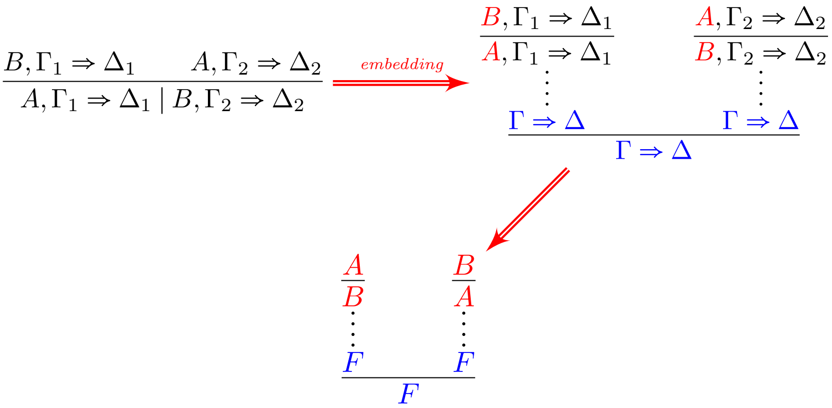
(The embedding does not introduce new applications of the cut rule)

Hypersequents made natural

The style of systems of rules
has a strong Natural Deduction (N.D.) flavour

Hypersequents made natural

Hence, the embedding provides a connection between the hypersequent calculus and N.D.



N.D. and λ-calculus for Gödel Logic

$$\dfrac{\begin{array}{ccc} \dfrac{B}{A} & & \dfrac{A}{B} \\ \vdots & & \vdots \\ F & & F \end{array}}{F}$$

[Aschieri, Ciabattoni and Genco. Submitted.]

- Standard rules
- Normalisation procedure
- Subformula property

Computational interpretations for hypersequents

Find **computational interpretations** of logics
that are formalised by hypersequent calculi

(Long-standing open problem [Avron, 1991])

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(Long-standing open problem [Avron, 1991])

$$\begin{array}{ccccc}
\text{Logic} & \Longrightarrow & \text{Hypersequents} & \xrightarrow{\textit{Embedding}} & \text{Systems of Rules} \\
\Uparrow & & & & \Downarrow \\
\lambda\text{-Calculus} & & \xleftarrow{\textit{Curry–Howard Correspondence}} & & \text{N.D.}
\end{array}$$

Beyond hypersequents

All axiomatisable propositional intermediate logics are defined by ***canonical formulae*** [Chagrov and Zakharyaschev, 1997] which are **just outside the scope of hypersequents**

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All axiomatisable propositional intermediate logics are defined by ***canonical formulae*** [Chagrov and Zakharyaschev, 1997] which are **just outside the scope of hypersequents**

Three-level **systems of rules** seem a very promising option

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