A Cut-free Sequent Calculus for the Logic of Subset Spaces

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The logic of subset spaces

A. Dabrowski, L. Moss and R. Parikh (1996) introduced the bimodal logic of subset spaces (SSL) and its extension topologic which yields a refinement of Tarski's and McKinsey's topological interpretation of S4.

Subset frames: $\mathcal{X} = (X, \mathcal{O})$ where $\mathcal{O} \subseteq \mathcal{P}(X) \setminus \emptyset$.
A world (x, u) consists of a "point" x and an "open" $u \ni x$.

Modality K (and dual L): quantify over points in the same open u

Modality $\Box$ (and dual $\Diamond$): quantify over the worlds obtained by shrinking the environment of x

Some extensions of the language of (w)SSL:
overlap operator (Heinemann AiML 2006)
announcement operators (Balbiani, van Ditmarsch, Kudinov 2013)

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Hilbert system for SSL

- all substitution instances of tautologies of propositional logic
- $\Box(A \to B) \to (\Box A \to \Box B)$
- $\Box A \to (A \wedge \Box\Box A)$

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- $\mathrm{K}(A \to B) \to (\mathrm{K}A \to \mathrm{K}B)$
- $\mathrm{K}A \to (A \wedge \mathrm{KK}A)$
- $\mathrm{L}A \to \mathrm{KL}A$

} S5

- Persistence
  $(P \to \Box P) \wedge (\neg P \to \Box\neg P)$ for propositional atoms P
- Cross axiom
  $\mathrm{K}\Box A \to \Box\mathrm{K}A$

Rules: modus ponens, necessitation $A \vdash \Box A$ and $A \vdash \mathrm{K}A$

DMP'96: sound and complete w.r.t. subset spaces

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Labelled sequent calculus for SSL

Here we will present a cut-free labelled sequent calculus.

We proceed as follows:

- We identify a set of crucial properties of subset spaces and generalise to abstract subset spaces.
- We follow the method for defining labelled calculi but we use pairs of simple labels as terms for worlds in the definition of the calculus **LSSL-p**.
- We show proof-theoretic properties of this system, in particular the admissibility of cut.
- We present (sketch) a proof a completeness.

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From (concrete) subset spaces to properties

Let $\mathcal{X} = (X, \mathcal{O})$ be a subset frame.

Worlds are pairs (x, u) so that $x \in u$.
Generalisation: Introduce $\mathcal{W} \subseteq X \times \mathcal{O}$ to determine the worlds.

Every world (y, u) is K, L-accessible from all worlds (x, u).

World (x, v) is $\Box, \Diamond$-accessible from (x, u) if $u \supseteq v$.
Generalisation: Use a relation $\mathcal{R} \subseteq \mathcal{O} \times \mathcal{O}$ for the condition on u, v.

Some properties of $\in, \supseteq$:

- $\supseteq$ is reflexive and transitive.
- If $x \in v$ and $u \supseteq v$ then $x \in u$.

$\mathcal{V}(x, u) = \mathcal{V}(x, v)$ for valuations $\mathcal{V}$

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Abstract subset spaces

Definition

An abstract subset frame $(X, \mathcal{O}, \mathcal{W}, \mathcal{R})$ consists of sets $X, \mathcal{O}$, a relation $\mathcal{W} \subseteq X \times \mathcal{O}$ and a preorder $\mathcal{R} \subseteq \mathcal{O} \times \mathcal{O}$ so that $\mathcal{W}; \mathcal{R}^{-1} \subseteq \mathcal{W}$.
An abstract subset space $(X, \mathcal{O}, \mathcal{W}, \mathcal{R}, \mathcal{V})$ consists of an abstract subset frame $(X, \mathcal{O}, \mathcal{W}, \mathcal{R})$ and a valuation $\mathcal{V} : X \to (\mathrm{PV} \to \mathbb{B})$.

Lemma

Let $X, \mathcal{O}$ be sets and $\mathcal{W}_0, \mathcal{R}_0$ relations so that $\mathcal{W}_0 \subseteq X \times \mathcal{O}$ and $\mathcal{R}_0 \subseteq \mathcal{O} \times \mathcal{O}$. Then $(X, \mathcal{O}, (\mathcal{W}_0; (\mathcal{R}_0^*)^{-1}), \mathcal{R}_0^*)$ is the least abstract subset frame $(X, \mathcal{O}, \mathcal{W}, \mathcal{R})$ so that $\mathcal{W}_0 \subseteq \mathcal{W}$ and $\mathcal{R}_0 \subseteq \mathcal{R}$.

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Abstract subset spaces as Kripke models

Definition

Let $(X, \mathcal{O}, \mathcal{W}, \mathcal{R}, \mathcal{V})$ be an abstract subset space. Then let:

$\hat{\mathcal{S}} := \{((x, u), (y, u)) \mid (x, u), (y, u) \in \mathcal{W}\}$

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Cross axiom model (DMP'96): $(\mathcal{W}, \mathcal{S}, \mathcal{R}, \mathcal{V})$ where $\mathcal{S}$ is an equivalence relation on a set $\mathcal{W}$, and $\mathcal{R}$ is a preorder on $\mathcal{W}$ so that $\mathcal{R}; \mathcal{S} \subseteq \mathcal{S}; \mathcal{R}$. Furthermore, $\mathcal{V} : \mathcal{W} \to (\mathrm{PV} \to \mathbb{B})$ is a valuation so that $\mathcal{V}(w) = \mathcal{V}(w')$ whenever $(w, w') \in \mathcal{R}$.

Lemma

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One-sided labelled sequent calculi

One-sided sequents: multisets of formulas in NNF

Propositional GS3 (Gentzen-Schütte system) :

$$\frac{}{\Gamma, P, \neg P} \qquad \frac{\Gamma, A \qquad \Gamma, B}{\Gamma, A \wedge B} \qquad \frac{\Gamma, A, B}{\Gamma, A \vee B}$$

Labelled calculi for modal logic (Negri '05): method for generating cut-free sequent calculi for a large variety of modal logics, e.g. S4 (originally two-sided, rewritten in one-sided style):

$$\frac{\Gamma, x\,\overline{R}\,y, y : A}{\Gamma, x : \Box A}\,!(y) \qquad \frac{\Gamma, x\,\overline{R}\,y, x : \Diamond A, y : A}{\Gamma, x\,\overline{R}\,y, x : \Diamond A}$$

$$\frac{x\,\overline{R}\,x, \Gamma}{\Gamma}\,\text{Reflexivity} \qquad \frac{x\,\overline{R}\,z, x\,\overline{R}\,y, y\,\overline{R}\,z, \Gamma}{x\,\overline{R}\,y, y\,\overline{R}\,z, \Gamma}\,\text{Transitivity}$$

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$$\frac{x\overline{R}x, \Gamma}{\Gamma} \text{ Reflexivity} \qquad \frac{x\overline{R}z, x\overline{R}y, y\overline{R}z, \Gamma}{x\overline{R}y, y\overline{R}z, \Gamma} \text{ Transitivity}$$

LSSL-p: judgements, sequents, side conditions

Let L_1, L_2 be two disjoint sets of labels.

Judgements: (for $x \in L_1,\ u, v \in L_2$)

$x\,\overline{W}\,u$	"(x, u) is no world"
$u\,\overline{R}\,v$	"v is not accessible from u"
$(x, u) \colon A$	"if (x, u) is a world then formula A holds at (x, u)"

Sequents are multisets of judgements

Side conditions for the application of rules:
$!(x)$ or $!(u)$: the label x/u does not occur in the conclusion
$j(x, u)$: the conclusion contains some $(x, u) : A$ or $x\,\overline{W}\,u$

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Basic properties

Let $\vdash$ (and $\vdash^n$) denote derivability in **LSSL-p** (with height $\leq n$).

Lemma

The following holds for **LSSL-p***:*

1. *(weakening)* $\vdash^n \Gamma \Longrightarrow \vdash^n \Gamma, J$ *for every judgement* J
2. *(R-contraction)* $\vdash^n \Gamma, u\,\overline{R}\,v, u\,\overline{R}\,v \Longrightarrow \vdash^n \Gamma, u\,\overline{R}\,v$
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Invertibility of logical rules

Theorem (Invertibility of logical rules)

The rules $\wedge$, $\vee$, $\Box$ and K are (height-preserving) invertible, i.e.:

1. $\vdash^n \Gamma, (x,u) : A \wedge B \Longrightarrow \vdash^n \Gamma, (x,u) : A$ and $\vdash^n \Gamma, (x,u) : B$
2. $\vdash^n \Gamma, (x,u) : A \vee B \Longrightarrow \vdash^n \Gamma, (x,u) : A, (x,u) : B$
3. $\vdash^n \Gamma, (x,u) : \Box A \Longrightarrow \vdash^n \Gamma, u\,\overline{R}\,v, x\,\overline{W}\,u, (x,v) : A$ for all $v \in L_2$
4. $\vdash^n \Gamma, (x,u) : \mathrm{K} A \Longrightarrow \vdash^n \Gamma, x\,\overline{W}\,u, (y,u) : A$ for all $y \in L_1$

Theorem (Admissibility of contraction)

If $\vdash^n \Gamma, (x,u) : A, (x,u) : A$ *then* $\vdash^n \Gamma, (x,u) : A$

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Admissibility of cut

Lemma

If $\vdash^n \Gamma\,,\,(x,u)\colon A$ *and* $\vdash^m \Pi\,,\,(x,u)\colon \neg A$*, then* $\vdash \Gamma, \Pi\,,\, x\,\overline{W}\,u$.

Corollary

Let Γ, Π *be sequents so that* Γ, Π *contains* $x\,\overline{W}\,u$ *or some judgement of the form* $(x,u)\colon B$.

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If **HSS** $\vdash A$ *then* **LSSL-p** $\vdash (x,u)\colon A$ *for all* $x \in L_1, u \in L_2$.

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Closure operators

$$\mathbf{L}_1(\Gamma) = \{x \in L_1 \mid x \text{ occurs in } \Gamma\}, \mathbf{L}_2(\Gamma) = \{u \in L_2 \mid u \text{ occurs in } \Gamma\}$$

$$\mathbf{R}_0(\Gamma) = \{(u,v) \mid u \, \overline{R} \, v \text{ occurs in } \Gamma\} \subseteq \mathbf{L}_2(\Gamma) \times \mathbf{L}_2(\Gamma)$$

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$$\mathbf{R}(\Gamma) = \mathbf{R}_0(\Gamma)^* \subseteq \mathbf{L}_2(\Gamma) \times \mathbf{L}_2(\Gamma)$$

$$\mathbf{W}(\Gamma) = \mathbf{W}_0(\Gamma); \mathbf{R}(\Gamma)^{-1}$$

for (finite or infinite) multisets Γ of judgements.

Let $\mathbf{D}(\Gamma)$ denote the least multiset extending Γ that satisfies:

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- $((x,v) \colon A) \in \mathbf{D}(\Gamma)$ if $(x,v) \in \mathbf{W}(\Gamma)$ and $((x,u) \colon \Diamond A) \in \mathbf{D}(\Gamma)$ for some u so that $(u,v) \in \mathbf{R}(\Gamma)$
- $((y,u) \colon A) \in \mathbf{D}(\Gamma)$ if $(y,u) \in \mathbf{W}(\Gamma)$ and $((x,u) \colon \mathrm{L}A) \in \mathbf{D}(\Gamma)$ for some x

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Compressed version of the calculus

$$(\mathrm{ax})\ \frac{}{\Gamma, (x,u) \colon P, (x,v) \colon \neg P}$$

$$(\wedge)\ \frac{\Gamma, (x,u) \colon A \qquad \Gamma, (x,u) \colon B}{\Gamma, (x,u) \colon A \wedge B} \qquad (\mathbf{D})\ \frac{\mathbf{D}(\Gamma)}{\Gamma}$$

$$(\Box)\ \frac{\Gamma, x\,\overline{W}\,u, u\,\overline{R}\,v, (x,v) \colon A}{\Gamma, (x,u) \colon \Box A}\ !(v) \qquad (\mathrm{K})\ \frac{\Gamma, x\,\overline{W}\,u, (y,u) \colon A}{\Gamma, (x,u) \colon \mathrm{K} A}\ !(y)$$

Proof search for **LSSL-pc** can be performed in such a way that

- either the search terminates producing a derivation of the goal formula
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Conclusion

- We formulated a cut-free calculus for SSL with good proof-theoretic properties
- To achieve this, we used Negri's method for constructing labelled sequent systems for modal logics.
- Introducing a new variant we obtained a system with conveniently simple rules.

Further work:

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