Fully Arbitrary Public Announcements

Hans van Ditmarsch[1], Wiebe van der Hoek[2] and Louwe B. Kuijer[1,2]

[1]University of Liverpool, [2]LORIA, CNRS

Table of Contents

Epistemic Logic

Start with: Epistemic Logic.

$$\varphi ::= p \mid \neg\varphi \mid \varphi \vee \varphi \mid K_a\varphi$$

Interpretation: $K_a\varphi$ means: agent a know that φ is true.
(Dual $\hat{K}_a\varphi$: agent a thinks φ might be true.)

Epistemic Logic

Start with: Epistemic Logic.

$$\varphi ::= p \mid \neg\varphi \mid \varphi \vee \varphi \mid K_a\varphi$$

Interpretation: $K_a\varphi$ means: agent a know that φ is true.
(Dual $\hat{K}_a\varphi$: agent a thinks φ might be true.)

Epistemic Logic

Start with: Epistemic Logic.

$$\varphi ::= p \mid \neg\varphi \mid \varphi \vee \varphi \mid K_a\varphi$$

Interpretation: $K_a\varphi$ means: agent a know that φ is true.
(Dual $\hat{K}_a\varphi$: agent a thinks φ might be true.)

Public Announcement Logic (I)

Add: public announcement operator. [Plaza, 1989][Baltag et al., 1998]

$$\varphi ::= p \mid \neg\varphi \mid \varphi \vee \varphi \mid K_a\varphi \mid [\varphi]\varphi$$

$[\psi]\varphi$ means: if ψ is truthfully and publicly announced, then φ will be true.

Public Announcement Logic (I)

Add: public announcement operator. [Plaza, 1989][Baltag et al., 1998]

$$\varphi ::= p \mid \neg\varphi \mid \varphi \vee \varphi \mid K_a\varphi \mid [\varphi]\varphi$$

$[\psi]\varphi$ means: if ψ is truthfully and publicly announced, then φ will be true.

Public Announcement Logic (I)

Add: public announcement operator. [Plaza, 1989][Baltag et al., 1998]

$$\varphi ::= p \mid \neg\varphi \mid \varphi \vee \varphi \mid K_a\varphi \mid [\varphi]\varphi$$

$[\psi]\varphi$ means: if ψ is truthfully and publicly announced, then φ will be true.

Public Announcement Logic (II)

Example: card game. Alice holds 7 of spades, Bob holds king of clubs, Claire holds ace of hearts.
Notation: $a : 7\spadesuit$, $b : K\clubsuit$, $c : A\heartsuit$.

I am an observer, see all cards.

I say out loud: Claire holds $7\spadesuit$ or $A\heartsuit$.

Result: Alice knows that Claire holds $A\heartsuit$, Bob does not.

In formulas: $[(c : 7\spadesuit) \vee (c : A\heartsuit)](K_a(c : A\heartsuit) \wedge \neg K_b(c : A\heartsuit))$

Public Announcement Logic (II)

Example: card game. Alice holds 7 of spades, Bob holds king of clubs, Claire holds ace of hearts.
Notation: $a : 7\spadesuit$, $b : K\clubsuit$, $c : A\heartsuit$.

I am an observer, see all cards.

I say out loud: Claire holds $7\spadesuit$ or $A\heartsuit$.

Result: Alice knows that Claire holds $A\heartsuit$, Bob does not.

In formulas: $[(c : 7\spadesuit) \vee (c : A\heartsuit)](K_a(c : A\heartsuit) \wedge \neg K_b(c : A\heartsuit))$

Public Announcement Logic (II)

Example: card game. Alice holds 7 of spades, Bob holds king of clubs, Claire holds ace of hearts.
Notation: $a : 7\spadesuit$, $b : K\clubsuit$, $c : A\heartsuit$.

I am an observer, see all cards.

I say out loud: Claire holds $7\spadesuit$ or $A\heartsuit$.

Result: Alice knows that Claire holds $A\heartsuit$, Bob does not.

In formulas: $[(c : 7\spadesuit) \vee (c : A\heartsuit)](K_a(c : A\heartsuit) \wedge \neg K_b(c : A\heartsuit))$

Public Announcement Logic (II)

Example: card game. Alice holds 7 of spades, Bob holds king of clubs, Claire holds ace of hearts.
Notation: $a : 7\spadesuit$, $b : K\clubsuit$, $c : A\heartsuit$.

I am an observer, see all cards.

I say out loud: Claire holds $7\spadesuit$ or $A\heartsuit$.

Result: Alice knows that Claire holds $A\heartsuit$, Bob does not.

In formulas: $[(c : 7\spadesuit) \vee (c : A\heartsuit)](K_a(c : A\heartsuit) \wedge \neg K_b(c : A\heartsuit))$

Public Announcement Logic (II)

Example: card game. Alice holds 7 of spades, Bob holds king of clubs, Claire holds ace of hearts.
Notation: $a : 7\spadesuit$, $b : K\clubsuit$, $c : A\heartsuit$.

I am an observer, see all cards.

I say out loud: Claire holds $7\spadesuit$ or $A\heartsuit$.

Result: Alice knows that Claire holds $A\heartsuit$, Bob does not.

In formulas: $[(c : 7\spadesuit) \vee (c : A\heartsuit)](K_a(c : A\heartsuit) \wedge \neg K_b(c : A\heartsuit))$

Arbitrary Public Announcement Logic (I)

Add: arbitrary public announcement operator $\Box$. [Balbiani et al., 2007]

$$\varphi ::= p \mid \neg\varphi \mid \varphi \vee \varphi \mid K_a\varphi \mid [\varphi]\varphi \mid \Box\varphi$$

$\Box\varphi$ means: for every ψ, we have $[\psi]\varphi$.
Dual: $\Diamond\varphi$ means: for some ψ, we have $[\psi]\varphi$ and ψ is true.

Arbitrary Public Announcement Logic (I)

Add: arbitrary public announcement operator $\Box$. [Balbiani et al., 2007]

$$\varphi ::= p \mid \neg\varphi \mid \varphi \vee \varphi \mid K_a\varphi \mid [\varphi]\varphi \mid \Box\varphi$$

$\Box\varphi$ means: for every ψ, we have $[\psi]\varphi$.
Dual: $\Diamond\varphi$ means: for some ψ, we have $[\psi]\varphi$ and ψ is true.

Arbitrary Public Announcement Logic (I)

Add: arbitrary public announcement operator $\Box$. [Balbiani et al., 2007]

$$\varphi ::= p \mid \neg\varphi \mid \varphi \vee \varphi \mid K_a\varphi \mid [\varphi]\varphi \mid \Box\varphi$$

$\Box\varphi$ means: for every ψ, we have $[\psi]\varphi$.
Dual: $\Diamond\varphi$ means: for some ψ, we have $[\psi]\varphi$ and ψ is true.

Arbitrary Public Announcement Logic (I)

Add: arbitrary public announcement operator $\Box$. [Balbiani et al., 2007]

$$\varphi ::= p \mid \neg\varphi \mid \varphi \vee \varphi \mid K_a\varphi \mid [\varphi]\varphi \mid \Box\varphi$$

$\Box\varphi$ means: for every ψ, we have $[\psi]\varphi$.
Dual: $\Diamond\varphi$ means: for some ψ, we have $[\psi]\varphi$ and ψ is true.

Arbitrary Public Announcement Logic (I)

Add: arbitrary public announcement operator $\Box$. [Balbiani et al., 2007]

$$\varphi ::= p \mid \neg\varphi \mid \varphi \vee \varphi \mid K_a\varphi \mid [\varphi]\varphi \mid \Box\varphi$$

$\Box\varphi$ means: for every ψ, we have $[\psi]\varphi$.
Dual: $\Diamond\varphi$ means: for some ψ, we have $[\psi]\varphi$ and ψ is true.

Arbitrary Public Announcement Logic (II)

Example: $\Diamond(K_c(a : 7\spadesuit) \land K_a K_c(a : 7\spadesuit) \land \neg K_b K_c(a : 7\spadesuit))$

Interpretation: there is something I could say that would result in (i) Claire knowing that Alice holds 7♠, (ii) Alice knowing that Claire knows and (iii) Bob not knowing that Claire knows.

I LIED

A little.

I LIED

A little.

APAL (the truth, this time)

Intuitive, desired meaning of $\Box\varphi$:

for every ψ, we have $[\psi]\varphi$.

Technical meaning of $\Box\varphi$:

for every ψ *that does not contain* $\Box$, we have $[\psi]\varphi$.

Reason: excluding ψ that contain $\Box$ avoids circularity.
See [Balbiani et al., 2007] for details.

APAL (the truth, this time)

Intuitive, desired meaning of $\Box\varphi$:

for every ψ, we have $[\psi]\varphi$.

Technical meaning of $\Box\varphi$:

for every ψ *that does not contain* $\Box$, we have $[\psi]\varphi$.

Reason: excluding ψ that contain $\Box$ avoids circularity.
See [Balbiani et al., 2007] for details.

APAL (the truth, this time)

Intuitive, desired meaning of $\Box\varphi$:

for every ψ, we have $[\psi]\varphi$.

Technical meaning of $\Box\varphi$:

for every ψ *that does not contain* $\Box$, we have $[\psi]\varphi$.

Reason: excluding ψ that contain $\Box$ avoids circularity.
See [Balbiani et al., 2007] for details.

APAL (the truth, this time)

Intuitive, desired meaning of $\Box\varphi$:

for every ψ, we have $[\psi]\varphi$.

Technical meaning of $\Box\varphi$:

for every ψ *that does not contain* $\Box$, we have $[\psi]\varphi$.

Reason: excluding ψ that contain $\Box$ avoids circularity.
See [Balbiani et al., 2007] for details.

The 'intuitive' version for the semantics of $\Box\varphi$ more properly corresponds to its intended meaning 'φ is true after arbitrary announcements'. This version is not well-defined, as $\Box\varphi$ is itself one such announcement.

[Balbiani et al., 2007]

APAL example (revisited, I)

Consider this conversation:

Me, speaking out loud: There is something I could say, that would result in Alice learning that Claire holds the 9 of spades or the ace of hearts, without Bob finding out.

Claire, thinking to herself: Oh, then Alice must have the 7 of spades.

Alice, thinking to herself: Oh, then Claire must know that I have the 7 of spades.

Bob, thinking to himself: That tells me nothing.

APAL example (revisited, I)

Consider this conversation:

Me, speaking out loud: There is something I could say, that would result in Alice learning that Claire holds the 9 of spades or the ace of hearts, without Bob finding out.

Claire, thinking to herself: Oh, then Alice must have the 7 of spades.

Alice, thinking to herself: Oh, then Claire must know that I have the 7 of spades.

Bob, thinking to himself: That tells me nothing.

APAL example (revisited, I)

Consider this conversation:

Me, speaking out loud: There is something I could say, that would result in Alice learning that Claire holds the 9 of spades or the ace of hearts, without Bob finding out.

Claire, thinking to herself: Oh, then Alice must have the 7 of spades.

Alice, thinking to herself: Oh, then Claire must know that I have the 7 of spades.

Bob, thinking to himself: That tells me nothing.

APAL example (revisited, I)

Consider this conversation:

Me, speaking out loud: There is something I could say, that would result in Alice learning that Claire holds the 9 of spades or the ace of hearts, without Bob finding out.

Claire, thinking to herself: Oh, then Alice must have the 7 of spades.

Alice, thinking to herself: Oh, then Claire must know that I have the 7 of spades.

Bob, thinking to himself: That tells me nothing.

APAL example (revisited, II)

Let $\chi = (K_c(a : 7\spadesuit) \wedge K_a K_c(a : 7\spadesuit) \wedge \neg K_b K_c(a : 7\spadesuit))$.
Then: there is a true announcement ψ such that $[\psi]\chi$.

So, intuitively, we *should* have $\Diamond\chi$.

But: not guaranteed in APAL, since ψ contains $\Diamond$.

APAL example (revisited, II)

Let $\chi = (K_c(a : 7\spadesuit) \land K_a K_c(a : 7\spadesuit) \land \neg K_b K_c(a : 7\spadesuit))$.
Then: there is a true announcement ψ such that $[\psi]\chi$.

So, intuitively, we *should* have $\Diamond\chi$.

But: not guaranteed in APAL, since ψ contains $\Diamond$.

APAL example (revisited, II)

Let $\chi = (K_c(a:7\spadesuit) \wedge K_aK_c(a:7\spadesuit) \wedge \neg K_bK_c(a:7\spadesuit))$.
Then: there is a true announcement ψ such that $[\psi]\chi$.

So, intuitively, we *should* have $\Diamond\chi$.

But: not guaranteed in APAL, since ψ contains $\Diamond$.

Table of Contents

F-APAL: goal

Introducing: Fully Arbitrary Public Announcement Logic (F-APAL).

Goal: $\Box\varphi$ if and only if $[\psi]\varphi$ for every ψ.

Not lying this time. I mean *every* ψ,[1] whether it contains $\Box$ or not.

[1] More precise: every ψ in the relevant language (that of F-APAL).

F-APAL: goal

Introducing: Fully Arbitrary Public Announcement Logic (F-APAL).

Goal: $\Box\varphi$ if and only if $[\psi]\varphi$ for every ψ.

Not lying this time. I mean *every* ψ,[1] whether it contains $\Box$ or not.

[1] More precise: every ψ in the relevant language (that of F-APAL).

F-APAL: goal

Introducing: Fully Arbitrary Public Announcement Logic (F-APAL).

Goal: $\Box\varphi$ if and only if $[\psi]\varphi$ for every ψ.

Not lying this time. I mean *every* ψ,[1] whether it contains $\Box$ or not.

[1] More precise: every ψ in the relevant language (that of F-APAL).

F-APAL: goal

Introducing: Fully Arbitrary Public Announcement Logic (F-APAL).

Goal: $\Box\varphi$ if and only if $[\psi]\varphi$ for every ψ.

Not lying this time. I mean *every* ψ,[1] whether it contains $\Box$ or not.

[1] More precise: every ψ in the relevant language (that of F-APAL).

F-APAL: the cost (I)

We can succeed in this goal. F-APAL satisfies

$$\mathcal{M}, w \models \Box\varphi \text{ if and only if } \mathcal{M}, w \models [\psi]\varphi \text{ for all } \psi. \qquad (*)$$

But: at a high cost.

F-APAL uses auxiliary operators $\Box_\alpha$ for every ordinal α.

So F-APAL has a proper class of operators. :-(

F-APAL: the cost (I)

We can succeed in this goal. F-APAL satisfies

$$\mathcal{M}, w \models \Box\varphi \text{ if and only if } \mathcal{M}, w \models [\psi]\varphi \text{ for all } \psi. \qquad (*)$$

But: at a high cost.

F-APAL uses auxiliary operators $\Box_\alpha$ for every ordinal α.

So F-APAL has a proper class of operators. :-(

F-APAL: the cost (I)

We can succeed in this goal. F-APAL satisfies

$$\mathcal{M}, w \models \Box\varphi \text{ if and only if } \mathcal{M}, w \models [\psi]\varphi \text{ for all } \psi. \qquad (*)$$

But: at a high cost.

F-APAL uses auxiliary operators $\Box_\alpha$ for every ordinal α.

So F-APAL has a proper class of operators. :-(

F-APAL: the cost (I)

We can succeed in this goal. F-APAL satisfies

$$\mathcal{M}, w \models \Box\varphi \text{ if and only if } \mathcal{M}, w \models [\psi]\varphi \text{ for all } \psi. \qquad (*)$$

But: at a high cost.

F-APAL uses auxiliary operators $\Box_\alpha$ for every ordinal α.

So F-APAL has a proper class of operators. :-(

F-APAL: the cost (I)

We can succeed in this goal. F-APAL satisfies

$$\mathcal{M}, w \models \Box\varphi \text{ if and only if } \mathcal{M}, w \models [\psi]\varphi \text{ for all } \psi. \qquad (*)$$

But: at a high cost.

F-APAL uses auxiliary operators $\Box_\alpha$ for every ordinal α.

So F-APAL has a proper class of operators. :-(

F-APAL: the cost (I)

We can succeed in this goal. F-APAL satisfies

$$\mathcal{M}, w \models \Box\varphi \text{ if and only if } \mathcal{M}, w \models [\psi]\varphi \text{ for all } \psi. \qquad (*)$$

But: at a high cost.

F-APAL uses auxiliary operators $\Box_\alpha$ for every ordinal α.

So F-APAL has a proper class of operators. :-(

F-APAL: the cost (II)

Is (*) worth the price of a proper class of operators?

If computational complexity is an issue: probably not.

From a purely theoretical point of view: I think so
(but it is still a heavy price).

F-APAL: the cost (II)

Is (*) worth the price of a proper class of operators?

If computational complexity is an issue: probably not.

From a purely theoretical point of view: I think so
(but it is still a heavy price).

F-APAL: the cost (II)

Is (*) worth the price of a proper class of operators?

If computational complexity is an issue: probably not.

From a purely theoretical point of view: I think so
(but it is still a heavy price).

F-APAL: language (I)

Done with introductory remarks. Time for formal definitions!

Language $\mathcal{L}$ of F-APAL:

$$\varphi ::= p \mid \neg\varphi \mid \varphi \vee \varphi \mid K_a\varphi \mid [\varphi]\varphi \mid \Box_\alpha\varphi \mid \Box\varphi$$

Where: p a propositional variable, a an agent, α an ordinal.

F-APAL: language (I)

Done with introductory remarks. Time for formal definitions!
Language $\mathcal{L}$ of F-APAL:

$$\varphi ::= p \mid \neg\varphi \mid \varphi \vee \varphi \mid K_a\varphi \mid [\varphi]\varphi \mid \Box_\alpha\varphi \mid \Box\varphi$$

Where: p a propositional variable, a an agent, α an ordinal.

F-APAL: language (I)

Done with introductory remarks. Time for formal definitions!
Language $\mathcal{L}$ of F-APAL:

$$\varphi ::= p \mid \neg\varphi \mid \varphi \vee \varphi \mid K_a\varphi \mid [\varphi]\varphi \mid \Box_\alpha\varphi \mid \Box\varphi$$

Where: p a propositional variable, a an agent, α an ordinal.

F-APAL: language (II)

For ordinal α, language $\mathcal{L}_\alpha$:

$$\varphi ::= p \mid \neg\varphi \mid \varphi \vee \varphi \mid K_a\varphi \mid [\varphi]\varphi \mid \Box_\beta\varphi$$

Where: $\beta < \alpha$.

In other words:
$\mathcal{L}_\alpha$ is the fragment of $\mathcal{L}$ without $\Box$ and without $\Box_\gamma$ for $\gamma \geq \alpha$.

F-APAL: language (II)

For ordinal α, language $\mathcal{L}_\alpha$:

$$\varphi ::= p \mid \neg\varphi \mid \varphi \vee \varphi \mid K_a\varphi \mid [\varphi]\varphi \mid \Box_\beta\varphi$$

Where: $\beta < \alpha$.

In other words:
$\mathcal{L}_\alpha$ is the fragment of $\mathcal{L}$ without $\Box$ and without $\Box_\gamma$ for $\gamma \geq \alpha$.

F-APAL: language (II)

For ordinal α, language $\mathcal{L}_\alpha$:

$$\varphi ::= p \mid \neg\varphi \mid \varphi \vee \varphi \mid K_a\varphi \mid [\varphi]\varphi \mid \Box_\beta\varphi$$

Where: $\beta < \alpha$.

In other words:
$\mathcal{L}_\alpha$ is the fragment of $\mathcal{L}$ without $\Box$ and without $\Box_\gamma$ for $\gamma \geq \alpha$.

F-APAL: semantics (I)

Operator $\Box_0$ quantifies over all formula that do not contain $\Box$ or $\Box_\alpha$.

So:

$$\mathcal{M}, w \models \Box_0\varphi \Leftrightarrow \forall\psi \in \mathcal{L}_0 : \mathcal{M}, w \models [\psi]\varphi.$$

Operator $\Box_1$ additionally quantifies over formulas that contain $\Box_0$.

So:

$$\mathcal{M}, w \models \Box_1\varphi \Leftrightarrow \forall\psi \in \mathcal{L}_1 : \mathcal{M}, w \models [\psi]\varphi.$$

F-APAL: semantics (I)

Operator $\Box_0$ quantifies over all formula that do not contain $\Box$ or $\Box_\alpha$.

So:

$$\mathcal{M}, w \models \Box_0\varphi \Leftrightarrow \forall\psi \in \mathcal{L}_0 : \mathcal{M}, w \models [\psi]\varphi.$$

Operator $\Box_1$ additionally quantifies over formulas that contain $\Box_0$.

So:

$$\mathcal{M}, w \models \Box_1\varphi \Leftrightarrow \forall\psi \in \mathcal{L}_1 : \mathcal{M}, w \models [\psi]\varphi.$$

F-APAL: semantics (I)

Operator $\Box_0$ quantifies over all formula that do not contain $\Box$ or $\Box_\alpha$.

So:

$$\mathcal{M}, w \models \Box_0\varphi \Leftrightarrow \forall\psi \in \mathcal{L}_0 : \mathcal{M}, w \models [\psi]\varphi.$$

Operator $\Box_1$ additionally quantifies over formulas that contain $\Box_0$.

So:

$$\mathcal{M}, w \models \Box_1\varphi \Leftrightarrow \forall\psi \in \mathcal{L}_1 : \mathcal{M}, w \models [\psi]\varphi.$$

F-APAL: semantics (I)

Operator $\Box_0$ quantifies over all formula that do not contain $\Box$ or $\Box_\alpha$.

So:

$$\mathcal{M}, w \models \Box_0\varphi \Leftrightarrow \forall\psi \in \mathcal{L}_0 : \mathcal{M}, w \models [\psi]\varphi.$$

Operator $\Box_1$ additionally quantifies over formulas that contain $\Box_0$.

So:

$$\mathcal{M}, w \models \Box_1\varphi \Leftrightarrow \forall\psi \in \mathcal{L}_1 : \mathcal{M}, w \models [\psi]\varphi.$$

F-APAL: semantics (II)

In general:

$$\mathcal{M}, w \models \Box_\alpha \varphi \Leftrightarrow \forall \psi \in \mathcal{L}_\alpha : \mathcal{M}, w \models [\psi]\varphi.$$

Operator $\Box$: conjunction of $\Box_\alpha$ for all α.

$$\mathcal{M}, w \models \Box \varphi \Leftrightarrow \forall \alpha \in \mathrm{Ord} : \mathcal{M}, w \models \Box_\alpha \varphi.$$

F-APAL: semantics (II)

In general:

$$\mathcal{M}, w \models \Box_\alpha \varphi \Leftrightarrow \forall \psi \in \mathcal{L}_\alpha : \mathcal{M}, w \models [\psi]\varphi.$$

Operator $\Box$: conjunction of $\Box_\alpha$ for all α.

$$\mathcal{M}, w \models \Box \varphi \Leftrightarrow \forall \alpha \in \mathrm{Ord} : \mathcal{M}, w \models \Box_\alpha \varphi.$$

F-APAL: semantics (II)

In general:

$$\mathcal{M}, w \models \Box_\alpha \varphi \Leftrightarrow \forall \psi \in \mathcal{L}_\alpha : \mathcal{M}, w \models [\psi]\varphi.$$

Operator $\Box$: conjunction of $\Box_\alpha$ for all α.

$$\mathcal{M}, w \models \Box \varphi \Leftrightarrow \forall \alpha \in \mathrm{Ord} : \mathcal{M}, w \models \Box_\alpha \varphi.$$

F-APAL: semantics (II)

In general:

$$\mathcal{M}, w \models \Box_\alpha \varphi \Leftrightarrow \forall \psi \in \mathcal{L}_\alpha : \mathcal{M}, w \models [\psi]\varphi.$$

Operator $\Box$: conjunction of $\Box_\alpha$ for all α.

$$\mathcal{M}, w \models \Box \varphi \Leftrightarrow \forall \alpha \in \mathrm{Ord} : \mathcal{M}, w \models \Box_\alpha \varphi.$$

F-APAL: fully arbitrary? (I)

Semantics of $\Box$: well-founded, therefore well-defined,
i. e. for every $\mathcal{M}, w$ and every φ, exactly one of $\mathcal{M}, w \models \varphi$ and $\mathcal{M}, w \not\models \varphi$ consistent with definition.[2]

[2] At least: if we fix a set-theoretic universe.

F-APAL: fully arbitrary? (I)

Semantics of $\Box$: well-founded, therefore well-defined,
i. e. for every $\mathcal{M}, w$ and every φ, exactly one of $\mathcal{M}, w \models \varphi$ and $\mathcal{M}, w \not\models \varphi$ consistent with definition.[2]

[2] At least: if we fix a set-theoretic universe.

F-APAL: fully arbitrary? (I)

Semantics of $\Box$: well-founded, therefore well-defined,
i. e. for every $\mathcal{M}, w$ and every φ, exactly one of $\mathcal{M}, w \models \varphi$ and $\mathcal{M}, w \not\models \varphi$ consistent with definition.[2]

[2] At least: if we fix a set-theoretic universe.

F-APAL: fully arbitrary? (II)

But! Remember (*):

$$\mathcal{M}, w \models \Box\varphi \text{ if and only if } \mathcal{M}, w \models [\psi]\varphi \text{ for all } \psi. \qquad (*)$$

Compare with definition of $\Box$:

$$\mathcal{M}, w \models \Box\varphi \Leftrightarrow \forall\alpha \in \mathrm{Ord} : \mathcal{M}, w \models \Box_\alpha\varphi.$$

Not immediate from definition that $\Box$ satisfies (*)!

F-APAL: fully arbitrary? (II)

But! Remember (*):

$$\mathcal{M}, w \models \Box\varphi \text{ if and only if } \mathcal{M}, w \models [\psi]\varphi \text{ for all } \psi. \qquad (*)$$

Compare with definition of $\Box$:

$$\mathcal{M}, w \models \Box\varphi \Leftrightarrow \forall\alpha \in \mathrm{Ord} : \mathcal{M}, w \models \Box_\alpha\varphi.$$

Not immediate from definition that $\Box$ satisfies (*)!

F-APAL: fully arbitrary? (II)

But! Remember (*):

$$\mathcal{M}, w \models \Box\varphi \text{ if and only if } \mathcal{M}, w \models [\psi]\varphi \text{ for all } \psi. \qquad (*)$$

Compare with definition of $\Box$:

$$\mathcal{M}, w \models \Box\varphi \Leftrightarrow \forall\alpha \in \mathrm{Ord} : \mathcal{M}, w \models \Box_\alpha\varphi.$$

Not immediate from definition that $\Box$ satisfies (*)!

F-APAL: fully arbitrary! (I)

Not immediate but still true:

Theorem

$\Box$ *in F-APAL is a fully arbitrary public announcement, i.e. it satisfies* (*).

F-APAL: fully arbitrary! (II)

Proof sketch.

Fix any model $\mathcal{M} = (W, R, V)$. Let $E_\alpha = \{[\![\varphi]\!] \mid \varphi \in \mathcal{L}_\alpha\}$. The E_α form increasing sequence. Suppose $E_\alpha = E_{\alpha+1}$. Then $\Box_\alpha$ and $\Box_{\alpha+1}$ quantify over the same set. So $\Box_{\alpha+1}$ doesn't add anything. Therefore: $E_{\alpha+2} = E_{\alpha+1} = E_\alpha$. By induction: $E_\alpha = E_\beta$ for all $\beta \geq \alpha$. Only $|2^W|$ different extensions on $\mathcal{M}$. So: $E_\beta = E_\gamma$ for all $\beta, \gamma > |2^W|$.

Therefore: for all φ, $M \models \Box\varphi \leftrightarrow \Box_{(|2^W|+1)}\varphi$. By construction, $\Box$ quantifies over all $\Box$-free formulas. By the equivalence, every formula with $\Box$ is equivalent (on $\mathcal{M}$) to one without. So: for every ψ, $\Box$ quantifies over formula that is equivalent to ψ.

□

F-APAL: fully arbitrary! (II)

Proof sketch.

Fix any model $\mathcal{M} = (W, R, V)$. Let $E_\alpha = \{[\![\varphi]\!] \mid \varphi \in \mathcal{L}_\alpha\}$. The E_α form increasing sequence. Suppose $E_\alpha = E_{\alpha+1}$. Then $\Box_\alpha$ and $\Box_{\alpha+1}$ quantify over the same set. So $\Box_{\alpha+1}$ doesn't add anything. Therefore: $E_{\alpha+2} = E_{\alpha+1} = E_\alpha$. By induction: $E_\alpha = E_\beta$ for all $\beta \geq \alpha$. Only $|2^W|$ different extensions on $\mathcal{M}$. So: $E_\beta = E_\gamma$ for all $\beta, \gamma > |2^W|$.

Therefore: for all φ, $M \models \Box\varphi \leftrightarrow \Box_{(|2^W|+1)}\varphi$. By construction, $\Box$ quantifies over all $\Box$-free formulas. By the equivalence, every formula with $\Box$ is equivalent (on $\mathcal{M}$) to one without. So: for every ψ, $\Box$ quantifies over formula that is equivalent to ψ.

□

F-APAL: fully arbitrary! (II)

Proof sketch.

Fix any model $\mathcal{M} = (W, R, V)$. Let $E_\alpha = \{[\![\varphi]\!] \mid \varphi \in \mathcal{L}_\alpha\}$. The E_α form increasing sequence. Suppose $E_\alpha = E_{\alpha+1}$. Then $\Box_\alpha$ and $\Box_{\alpha+1}$ quantify over the same set. So $\Box_{\alpha+1}$ doesn't add anything. Therefore: $E_{\alpha+2} = E_{\alpha+1} = E_\alpha$. By induction: $E_\alpha = E_\beta$ for all $\beta \geq \alpha$.
Only $|2^W|$ different extensions on $\mathcal{M}$. So: $E_\beta = E_\gamma$ for all $\beta, \gamma > |2^W|$.

Therefore: for all φ, $M \models \Box\varphi \leftrightarrow \Box_{(|2^W|+1)}\varphi$. By construction, $\Box$ quantifies over all $\Box$-free formulas. By the equivalence, every formula with $\Box$ is equivalent (on $\mathcal{M}$) to one without. So: for every ψ, $\Box$ quantifies over formula that is equivalent to ψ.

□

F-APAL: fully arbitrary! (II)

Proof sketch.

Fix any model $\mathcal{M} = (W, R, V)$. Let $E_\alpha = \{[\![\varphi]\!] \mid \varphi \in \mathcal{L}_\alpha\}$. The E_α form increasing sequence. Suppose $E_\alpha = E_{\alpha+1}$. Then $\Box_\alpha$ and $\Box_{\alpha+1}$ quantify over the same set. So $\Box_{\alpha+1}$ doesn't add anything. Therefore: $E_{\alpha+2} = E_{\alpha+1} = E_\alpha$. By induction: $E_\alpha = E_\beta$ for all $\beta \geq \alpha$. Only $|2^W|$ different extensions on $\mathcal{M}$. So: $E_\beta = E_\gamma$ for all $\beta, \gamma > |2^W|$.

Therefore: for all φ, $M \models \Box\varphi \leftrightarrow \Box_{(|2^W|+1)}\varphi$. By construction, $\Box$ quantifies over all $\Box$-free formulas. By the equivalence, every formula with $\Box$ is equivalent (on $\mathcal{M}$) to one without. So: for every ψ, $\Box$ quantifies over formula that is equivalent to ψ.

□

F-APAL: fully arbitrary! (II)

Proof sketch.

Fix any model $\mathcal{M} = (W, R, V)$. Let $E_\alpha = \{[\![\varphi]\!] \mid \varphi \in \mathcal{L}_\alpha\}$. The E_α form increasing sequence. Suppose $E_\alpha = E_{\alpha+1}$. Then $\Box_\alpha$ and $\Box_{\alpha+1}$ quantify over the same set. So $\Box_{\alpha+1}$ doesn't add anything. Therefore: $E_{\alpha+2} = E_{\alpha+1} = E_\alpha$. By induction: $E_\alpha = E_\beta$ for all $\beta \geq \alpha$.
Only $|2^W|$ different extensions on $\mathcal{M}$. So: $E_\beta = E_\gamma$ for all $\beta, \gamma > |2^W|$.

Therefore: for all φ, $M \models \Box\varphi \leftrightarrow \Box_{(|2^W|+1)}\varphi$. By construction, $\Box$ quantifies over all $\Box$-free formulas. By the equivalence, every formula with $\Box$ is equivalent (on $\mathcal{M}$) to one without. So: for every ψ, $\Box$ quantifies over formula that is equivalent to ψ.

□

F-APAL: fully arbitrary! (II)

Proof sketch.

Fix any model $\mathcal{M} = (W, R, V)$. Let $E_\alpha = \{[\![\varphi]\!] \mid \varphi \in \mathcal{L}_\alpha\}$. The E_α form increasing sequence. Suppose $E_\alpha = E_{\alpha+1}$. Then $\Box_\alpha$ and $\Box_{\alpha+1}$ quantify over the same set. So $\Box_{\alpha+1}$ doesn't add anything. Therefore: $E_{\alpha+2} = E_{\alpha+1} = E_\alpha$. By induction: $E_\alpha = E_\beta$ for all $\beta \geq \alpha$.
Only $|2^W|$ different extensions on $\mathcal{M}$. So: $E_\beta = E_\gamma$ for all $\beta, \gamma > |2^W|$.

Therefore: for all φ, $M \models \Box\varphi \leftrightarrow \Box_{(|2^W|+1)}\varphi$. By construction, $\Box$ quantifies over all $\Box$-free formulas. By the equivalence, every formula with $\Box$ is equivalent (on $\mathcal{M}$) to one without. So: for every ψ, $\Box$ quantifies over formula that is equivalent to ψ.

□

F-APAL: fully arbitrary! (II)

Proof sketch.

Fix any model $\mathcal{M} = (W, R, V)$. Let $E_\alpha = \{[\![\varphi]\!] \mid \varphi \in \mathcal{L}_\alpha\}$. The E_α form increasing sequence. Suppose $E_\alpha = E_{\alpha+1}$. Then $\Box_\alpha$ and $\Box_{\alpha+1}$ quantify over the same set. So $\Box_{\alpha+1}$ doesn't add anything. Therefore: $E_{\alpha+2} = E_{\alpha+1} = E_\alpha$. By induction: $E_\alpha = E_\beta$ for all $\beta \geq \alpha$.
Only $|2^W|$ different extensions on $\mathcal{M}$. So: $E_\beta = E_\gamma$ for all $\beta, \gamma > |2^W|$.

Therefore: for all φ, $M \models \Box\varphi \leftrightarrow \Box_{(|2^W|+1)}\varphi$. By construction, $\Box$ quantifies over all $\Box$-free formulas. By the equivalence, every formula with $\Box$ is equivalent (on $\mathcal{M}$) to one without. So: for every ψ, $\Box$ quantifies over formula that is equivalent to ψ.

□

F-APAL: fully arbitrary! (II)

Proof sketch.

Fix any model $\mathcal{M} = (W, R, V)$. Let $E_\alpha = \{[\![\varphi]\!] \mid \varphi \in \mathcal{L}_\alpha\}$. The E_α form increasing sequence. Suppose $E_\alpha = E_{\alpha+1}$. Then $\Box_\alpha$ and $\Box_{\alpha+1}$ quantify over the same set. So $\Box_{\alpha+1}$ doesn't add anything. Therefore: $E_{\alpha+2} = E_{\alpha+1} = E_\alpha$. By induction: $E_\alpha = E_\beta$ for all $\beta \geq \alpha$.
Only $|2^W|$ different extensions on $\mathcal{M}$. So: $E_\beta = E_\gamma$ for all $\beta, \gamma > |2^W|$.

Therefore: for all φ, $M \models \Box\varphi \leftrightarrow \Box_{(|2^W|+1)}\varphi$. By construction, $\Box$ quantifies over all $\Box$-free formulas. By the equivalence, every formula with $\Box$ is equivalent (on $\mathcal{M}$) to one without. So: for every ψ, $\Box$ quantifies over formula that is equivalent to ψ.

□

Summary

In summary: $\Box$ in F-APAL is a fully arbitrary public announcement, satisfying

$$\mathcal{M}, w \models \Box\varphi \text{ if and only if } \mathcal{M}, w \models [\psi]\varphi \text{ for all } \psi. \qquad (*)$$

But at a price: F-APAL uses proper class of auxiliary operators $\Box_\alpha$.

Table of Contents

Avoiding the cost

F-APAL uses proper class of operators, conceptually expensive.
Can we avoid this cost, creating cheaper fully arbitrary public announcements?

Answer: we don't know, hard to prove non-existence of cheaper option.
But: salient easier alternatives fail.
We consider two such alternatives: ignoring the problem and fixed points.

Avoiding the cost

F-APAL uses proper class of operators, conceptually expensive.
Can we avoid this cost, creating cheaper fully arbitrary public announcements?

Answer: we don't know, hard to prove non-existence of cheaper option.
But: salient easier alternatives fail.
We consider two such alternatives: ignoring the problem and fixed points.

Avoiding the cost

F-APAL uses proper class of operators, conceptually expensive.
Can we avoid this cost, creating cheaper fully arbitrary public announcements?

Answer: we don't know, hard to prove non-existence of cheaper option.
But: salient easier alternatives fail.
We consider two such alternatives: ignoring the problem and fixed points.

Avoiding the cost

F-APAL uses proper class of operators, conceptually expensive.
Can we avoid this cost, creating cheaper fully arbitrary public announcements?

Answer: we don't know, hard to prove non-existence of cheaper option.
But: salient easier alternatives fail.
We consider two such alternatives: ignoring the problem and fixed points.

Avoiding the cost

F-APAL uses proper class of operators, conceptually expensive.
Can we avoid this cost, creating cheaper fully arbitrary public announcements?

Answer: we don't know, hard to prove non-existence of cheaper option.
But: salient easier alternatives fail.
We consider two such alternatives: ignoring the problem and fixed points.

Attempted solution 1: ignoring the problem

How about we just define

$$\mathcal{M}, w \models \Box\varphi \Leftrightarrow \forall\psi : \mathcal{M}, w \models [\psi]\varphi$$

Sure, that's circular. But maybe we are lucky and the circularity is non-vicious?
No such luck. :-(
This definition is viciously circular: it is underdetermined.

Attempted solution 1: ignoring the problem

How about we just define

$$\mathcal{M}, w \models \Box\varphi \Leftrightarrow \forall\psi : \mathcal{M}, w \models [\psi]\varphi$$

Sure, that's circular. But maybe we are lucky and the circularity is non-vicious?

No such luck. :-(

This definition is viciously circular: it is underdetermined.

Attempted solution 1: ignoring the problem

How about we just define

$$\mathcal{M}, w \models \Box\varphi \Leftrightarrow \forall\psi : \mathcal{M}, w \models [\psi]\varphi$$

Sure, that's circular. But maybe we are lucky and the circularity is non-vicious?
No such luck. :-(
This definition is viciously circular: it is underdetermined.

Attempted solution 1: ignoring the problem

How about we just define

$$\mathcal{M}, w \models \Box\varphi \Leftrightarrow \forall\psi : \mathcal{M}, w \models [\psi]\varphi$$

Sure, that's circular. But maybe we are lucky and the circularity is non-vicious?
No such luck. :-(
This definition is viciously circular: it is underdetermined.

Attempted solution 2: fixed points

Construction of $\Box$ as conjunction of all $\Box_\alpha$ resembles fixed point constructions. So maybe we can describe $\Box$ as a least fixed point?

Yes, we can describe $\Box$ as a fixed point. But: it is a fixed point of a non-monotone operator. So standard fixed point theorems don't apply. In particular: not known whether $\Box$ is a *least* fixed point. Also: auxiliary operators $\Box_\alpha$ still needed, so fixed point definition doesn't make things simpler.

Attempted solution 2: fixed points

Construction of $\Box$ as conjunction of all $\Box_\alpha$ resembles fixed point constructions. So maybe we can describe $\Box$ as a least fixed point?

Yes, we can describe $\Box$ as a fixed point. But: it is a fixed point of a non-monotone operator. So standard fixed point theorems don't apply. In particular: not known whether $\Box$ is a *least* fixed point.
Also: auxiliary operators $\Box_\alpha$ still needed, so fixed point definition doesn't make things simpler.

Attempted solution 2: fixed points

Construction of $\Box$ as conjunction of all $\Box_\alpha$ resembles fixed point constructions. So maybe we can describe $\Box$ as a least fixed point?

Yes, we can describe $\Box$ as a fixed point. But: it is a fixed point of a non-monotone operator. So standard fixed point theorems don't apply. In particular: not known whether $\Box$ is a *least* fixed point.
Also: auxiliary operators $\Box_\alpha$ still needed, so fixed point definition doesn't make things simpler.

Attempted solution 2: fixed points

Construction of $\Box$ as conjunction of all $\Box_\alpha$ resembles fixed point constructions. So maybe we can describe $\Box$ as a least fixed point?

Yes, we can describe $\Box$ as a fixed point. But: it is a fixed point of a non-monotone operator. So standard fixed point theorems don't apply.
In particular: not known whether $\Box$ is a *least* fixed point.
Also: auxiliary operators $\Box_\alpha$ still needed, so fixed point definition doesn't make things simpler.

Attempted solution 2: fixed points

Construction of $\Box$ as conjunction of all $\Box_\alpha$ resembles fixed point constructions. So maybe we can describe $\Box$ as a least fixed point?

Yes, we can describe $\Box$ as a fixed point. But: it is a fixed point of a non-monotone operator. So standard fixed point theorems don't apply. In particular: not known whether $\Box$ is a *least* fixed point.
Also: auxiliary operators $\Box_\alpha$ still needed, so fixed point definition doesn't make things simpler.

Attempted solution 2: fixed points

Construction of $\Box$ as conjunction of all $\Box_\alpha$ resembles fixed point constructions. So maybe we can describe $\Box$ as a least fixed point?

Yes, we can describe $\Box$ as a fixed point. But: it is a fixed point of a non-monotone operator. So standard fixed point theorems don't apply.
In particular: not known whether $\Box$ is a *least* fixed point.
Also: auxiliary operators $\Box_\alpha$ still needed, so fixed point definition doesn't make things simpler.

Avoiding the cost (II)

All in all: no obvious way to avoid the cost.

But: we are still searching.

Avoiding the cost (II)

All in all: no obvious way to avoid the cost.

But: we are still searching.

References

- Plaza, J., *Logics of public communication.* In: Emrich, M., Phifer, M., Hadzikadic, M., Ras, Z. (eds.) Proceedings of the Fourth International Symposium on Methodologies for Intelligent Systems, Poster Session Program, pp. 201216. 1989.
- Baltag, A., Moss, L., Solecki, S., *The logic of public announcements, common knowledge, and private suspicions.* In: Gilboa, I. (ed.) Proceedings of the 7th Conference on Theoretical Aspects of Rationality and Knowledge, pp. 4356. 1998.
- Balbiani, P., Baltag, A., van Ditmarsch, H., Herzig, A., Hoshi, T., de Lima, T. *What can we achieve by arbitrary announcements? A dynamic take on Fitchs knowability.* In: Samet, D. (ed.) Proceedings of the 11th Conference on Theoretical Aspects of Rationality and Knowledge, pp. 4251. 2007.