Towards a Herbrand's Theorem for Hybrid Logic

Diana Costa[1]
Manuel A. Martins[1]
João Marcos[2]

[1] CIDMA – Center for R&D in Mathematics and Applications
Department of Mathematics, University of Aveiro

2 Department of Informatics and Applied Mathematics
Federal University of Rio Grande do Norte

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A Herbrand's Theorem for Classical Logic

Theoretical basis for theorem proving.
Constructive method.
Reduction of first-order logic to propositional logic.

Theorem

χ has a first-order proof if and only if some χ_i is a tautology, where χ_i comes from a sequence of quantifier-free formulas $\chi_1, \chi_2, \chi_3, \cdots$ associated with χ.

What are Hybrid Logics?

Extension of propositional modal logic with the ability to refer to worlds by:

- considering a new class of atomic formulas, called nominals;
- using a new operator, @, called satisfaction operator.

A nominal is true at exactly one state: the one it names.

Definition

Let $\mathcal{L} = \langle \mathrm{Prop}, \mathrm{Nom} \rangle$ *be a hybrid similarity type.*

A hybrid structure $\mathcal{H}$ *over* $\mathcal{L}$ *is a tuple* (W, R, N, V)*, where:*

$W \neq \emptyset$ – *domain whose elements are called states or worlds,*

$R \subseteq W \times W$ – *accessibility relation,*

$N : \mathrm{Nom} \to W$ – *hybrid nomination,*

$V : \mathrm{Prop} \to Pow(W)$ – *hybrid valuation.*

Definition

For a hybrid similarity type $\mathcal{L} = \langle \mathrm{Prop}, \mathrm{Nom} \rangle$, *we define*

- **Atoms** *over* $\mathcal{L}$*:*

$\mathrm{At}(\mathcal{L}) = \{ @_i p,\ @_i j,\ @_i \Diamond j \mid i, j \in \mathrm{Nom}, p \in \mathrm{Prop} \}$;

- **Literals** *over* $\mathcal{L}$*:*

$\mathrm{Lit}(\mathcal{L}) = \{ @_i p,\ @_i \neg p,\ @_i j,\ @_i \neg j, @_i \Diamond j,\ @_i \Box \neg j \mid i, j \in \mathrm{Nom}, p \in \mathrm{Prop} \}$.

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Expansion of $\mathcal{L}$ by adding new nominals for the elements of the domain W: $\mathcal{L}(W) = \langle \mathrm{Prop}, \mathrm{Nom} \cup W \rangle$.

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Expansion of $\mathcal{L}$ by adding new nominals for the elements of the domain W: $\mathcal{L}(W) = \langle \mathrm{Prop}, \mathrm{Nom} \cup W \rangle$.

The diagram of a hybrid structure $\mathcal{H}$ over $\mathcal{L}$ is the set of literals over $\mathcal{L}(W)$ that are valid in $\mathcal{H}(W)$.
Given $\mathcal{L}$ and W, the diagram of $\mathcal{H}$ is unique.

Example

Let $\mathcal{L} = \langle\{p, q\}, \{\}\rangle$, *and* $W = \{u, v, w\}$

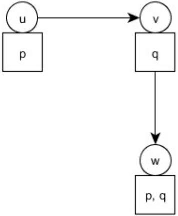

Figure: A hybrid structure.

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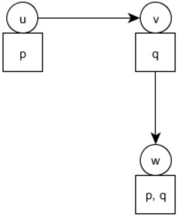

Figure: A hybrid structure.

$$diag(\mathcal{H}) = \{@_u p, @_u \neg q, @_v \neg p, @_v q, @_w p, @_w q$$

$$@_u \neg v, @_u \neg w, @_v \neg u, @_v \neg w, @_w \neg u, @_w \neg v$$

$$@_u \Diamond v, @_u \Box \neg u, @_u \Box \neg w, @_v \Diamond w, @_v \Box \neg u$$

$$@_v \Box \neg w, @_w \Box \neg u, @_w \Box \neg v, @_w \Box \neg w\}$$

Definition

Let $\mathcal{L}$ be a hybrid signature. We use $\mathrm{BCAt}(\mathcal{L})$ to denote the set of all (finite) Boolean combinations of atoms over $\mathcal{L}$, i.e., $\mathrm{BCAt}(\mathcal{L})$ is the smallest set containing $\mathrm{At}(\mathcal{L})$ and closed under $\wedge$ and $\neg$.

$\mathcal{L}$-truth assignment: $f : \mathrm{At}(\mathcal{L}) \to \{T, F\}$.
Extension to $\bar{f} : \mathrm{BCAt}(\mathcal{L}) \to \{T, F\}$.

Definition

Let $\Phi \subseteq \mathrm{BCAt}(\mathcal{L})$. We say that Φ is propositionally satisfiable if there is a truth assignment that simultaneously satisfies every member of Φ. We say that Φ is propositionally unsatisfiable if there is no such assignment.

First part of a Herbrand's theorem for hybrid logic:

Theorem

Let $\Phi \subseteq \mathrm{BCAt}(\mathcal{L})$. If Φ is propositionally unsatisfiable then Φ is unsatisfiable.

The converse of the previous theorem is not true in general; we have to consider equality axioms.

Hybrid formulas that express the equality axioms over nominals:

- Reflexivity: $@_i i$, for $i \in \mathrm{Nom}$;
- Symmetry: $@_i j \to @_j i$, for $i, j \in \mathrm{Nom}$;
- Transitivity: $(@_i j \wedge @_j k) \to @_i k$, for $i, j, k \in \mathrm{Nom}$;
- Congruence: $(@_i j \wedge @_k n) \to (@_i \Diamond k \leftrightarrow @_j \Diamond n)$, for $i, j, k, n \in \mathrm{Nom}$.

They will be denoted by $\mathrm{Eq}(\mathcal{L})$.

Theorem

Let $\Phi \subseteq \mathrm{BCAt}(\mathcal{L})$ *such that* $\mathrm{Eq}(\mathcal{L}) \subseteq \Phi$*. If* Φ *is unsatisfiable then* Φ *is propositionally unsatisfiable.*

Generalization for any Satisfaction Statement

Rules:

$$\frac{@_i \neg\varphi}{\neg @_i \varphi} \qquad \frac{@_i(\varphi \wedge \psi)}{@_i\varphi \wedge @_i\psi} \qquad \frac{@_i \Diamond\varphi}{@_i\Diamond k \wedge @_k\varphi}(*)$$

$(*)$ k is a fresh nominal

Theorem

Let Φ be a set of satisfaction statements such that $\mathrm{Eq}(\mathcal{L}) \subseteq \Phi$. Then Φ is propositionally unsatisfiable iff Φ is unsatisfiable.

First version of Herbrand's theorem in the context of Hybrid logic:

- with a restriction to satisfaction statements;
- where we transform each satisfaction statement into a boolean combination of atomic satisfaction statements;
- making use of the fact that each model can be described by its diagram [Bla00].

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- with a restriction to satisfaction statements;
- where we transform each satisfaction statement into a boolean combination of atomic satisfaction statements;
- making use of the fact that each model can be described by its diagram [Bla00].

A set of satisfaction statements is propositionally unsatisfiable if and only if it is unsatisfiable.

Next steps:

(1) Formulas with quantifiers constitute a challenge:

- To deal with quantifiers over world variables: add function symbols interpreted as functions on the set of worlds.
- Make use of translations between Priorean Hybrid Logic and First-Order Logic.
- Skolemization will occur in a standard way; a Herbrand's Theorem for PHL can be achieved.

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(2) Deal with First-Order Hybrid Logic.

- Patrick Blackburn, *Internalizing labelled deduction*, J. Log. Comput. **10** (2000), no. 1, 137–168.
- Evangelos Tzanis, *Algebraizing hybrid logic*, Master's thesis, University of Amsterdam - Institute of Logic, Language and Computation, 2005.