Temporal modal logics of intervals with the relation 'before'

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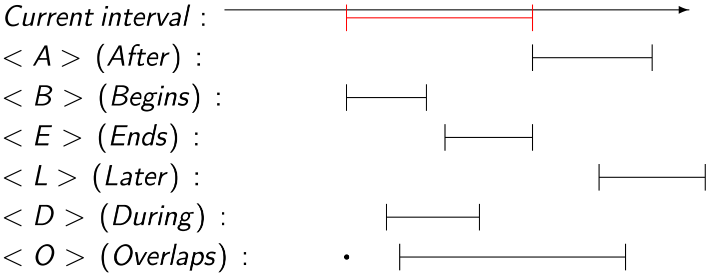
Introduction

We follow the definitions from ‘A propositional modal logic of time intervals’, J. Y. Halpern, Y. Shoham, 1991.
Recall that the Halpern-Shoham logic of intervals HS contains six modal operators. These operators and their converses define twelve possible relations between two distinct intervals. (See ‘Maintaining knowledge about temporal intervals’, J. Allen, 1983.)



The papers on HS-fragments

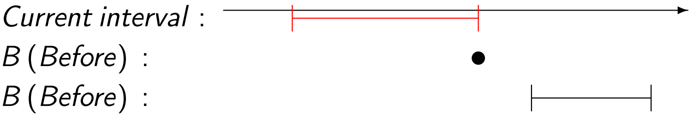
The HS logics of the intervals on $\mathbb{Q}$ and $\mathbb{R}$ are undecidable. Decidable fragments of HS-logics are rare, some of them are studied in the following works:

1. Chronological future modality in Minkowski spacetime, I. Shapirovsky, V. Shehtman, 2003, with the HS-modalities $< D >$ ('during') and the converse $< \overline{D} >$ ('contains').
2. Propositional interval neighborhood logics: expressiveness, decidability and undecidable extensions, D. Bresolin, V. Goranko, A. Montanari, G. Sciavicco, 2009, with the modality $< A >$ ('after') for strict and non-strict intervals in arbitrary linear order and in $\mathbb{Q}$.

The papers on HS-fragments

3. Propositional Interval Neighborhood Temporal Logics, V. Goranko, A. Montanari, G. Sciavicco, 2003 introduces Propositional Neighborhood Logic (PNL) with the modality $< A >$ ('after') and its converse. The work proposes finite axiomatization for PNL and proves its completeness for some cases (in particular, for $\mathbb{Z}$ and $\mathbb{Q}$).
4. An optimal tableau system for the logic of temporal neighborhood over the reals, A. Montanari, P. Sala, 2012, with the modality $< A >$ and its converse. NExptime completeness for satisfiability in PNL for $\mathbb{R}$ and $\mathbb{Q}$ are proved.
5. The Importance of the Past in Interval Temporal Logics: The Case of Propositional Neighborhood Logic, D. Monica, A. Montanari, P. Sala, 2012.

The main results



In this paper we consider non-strict intervals with the relation 'before', which can be expressed as 'after' ∘ 'after' (so the corresponding modality is <A><A>) and its converse. We obtain decidability and the finite model property (f.m.p.) of the corresponding temporal (bimodal) logics of closed intervals. The results are proved for the cases of $\mathbb{R}$ and $\mathbb{Q}$.

The unimodal case

The previous work, unimodal case:

'Modal logic of intervals with relation 'later' (in Russian),
A. Chizhov, 2016.

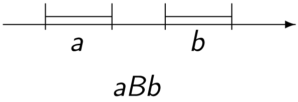
- Axiomatization;
- finite model property;
- the description of finite models;
- NP-completeness.

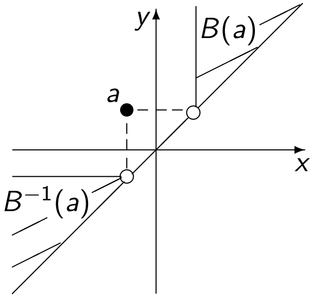
We hope that NP-completeness holds for the bimodal case.
Reason: the class of the finite frames corresponding to the temporal logic is a subclass of the finite frames corresponding to the unimodal logic.

Definitions

Let $\mathbb{I}_\mathbb{R}(\mathbb{I}_\mathbb{Q})$ be the temporal Kripke frame of non-strict intervals on the real (rational) numbers with the relation B (before) defined as follows:

$\underline{\mathbb{I}}_\mathbb{R} := \{(x, y) \mid x, y \in \mathbb{R}, x \leqslant y\}, \quad \underline{\mathbb{I}}_\mathbb{Q} := \{(x, y) \mid x, y \in \mathbb{Q}, x \leqslant y\}$
$(x_1, y_1)B(x_2, y_2) := y_1 \leqslant x_2,$
$\mathbb{I}_\mathbb{R} := (\underline{\mathbb{I}}_\mathbb{R}, B, B^{-1})$ where B^{-1} is the converse of B.





The temporal logic

We consider a standard bimodal language with the countable set of propositional variables and the modalities $\Box$ and $\Box^{-1}$.
Let φ^{-1} be the formula obtained from φ by changing all occurrences of $\Box$ to $\Box^{-1}$ and vice versa.

Axiomatization

We use the following axioms:

$A4 := \Diamond\Diamond p \to \Diamond p,$ $\qquad Ad_2 := \Diamond p \wedge \Diamond q \to \Diamond(\Diamond p \wedge \Diamond q),$

$AD := \Diamond\top,$ $\qquad ACL := \Box(\Box p \to \Box q) \vee \Box(\Box q \to \Box p),$

$ARP := \Diamond\Box^{-1}p \wedge \Diamond(\Box q \wedge \Diamond t) \to \Diamond(\Box^{-1}p \wedge \Box q \wedge \Diamond t),$

$$ASBC := \Box\Box^{-1}((\neg p \to \Box^{-1}q \wedge \Diamond(p \wedge \Diamond\neg p)) \wedge (\neg q \to \Box p \wedge \Diamond^{-1}(q \wedge \Diamond^{-1}\neg q)) \wedge (\Box p \wedge \Box^{-1}q \to \neg t \wedge \Box t)) \to \Box p \vee \Box q.$$

$L_0 := K + A4 + AD + Ad_2 + ACL,$

$L_1^t := K_t + A4 + AD + AD^{-1} + Ad_2 + Ad_2^{-1} + ACL + ACL^{-1} + ARP,$

$L_0^t := L_1^t + ASBC.$

The main results

<u>Theorem 1</u> The logic of $\mathbb{I}_{\mathbb{R}}$ is L_0^t.

<u>Theorem 1</u>$'$ The logic of $\mathbb{I}_{\mathbb{Q}}$ is L_1^t.

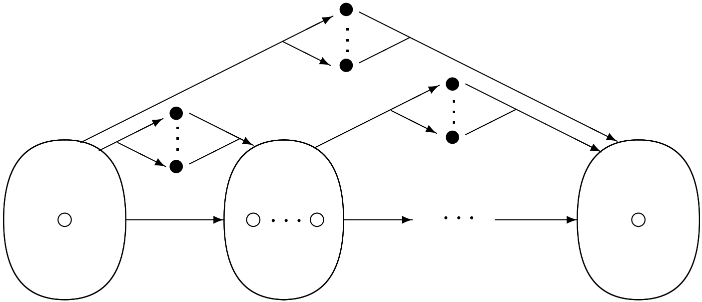
<u>Theorem 2</u> L_0^t and L_1^t have the finite model property.

<u>Theorem 3</u> L_0^t and L_1^t are decidable.

The case of $\mathbb{R}$ is much more difficult.

The finite models

<u>Proposition</u> Rooted finite L_0^t-frame F with the relation R looks as follows:



The finite models

Rooted finite L^t_0-frame F with the relation R satisfies the following conditions:
(i) the non-degenerate clusters are linearly ordered,
(ii) one of each two adjacent clusters is a singleton.
(iii) for every irrefexive t in F there are non-degenerate clusters C_1, C_2 such that $R^{-1}(t) = R^{-1}(C_1)$ and $R(t) = R(C_2)$.
(iv) for every pair of non-degenerate clusters $C_1 <_R C_2$ there is an irreflexive point t such that $R^{-1}(t) = R^{-1}(C_1)$ and $R(t) = R(C_2)$.

Rooted finite L^t_1-frames should satisfy only the conditions *(i)*, *(iii)*, *(iv)*.

Rooted finite L_0-frames should satisfy only the conditions *(i)*, *(iii)* and also may have an additional irreflexive root.

Proofs

Theorem 1 The logic of $\mathbb{I}_{\mathbb{R}}$ is L_0^t.

The proof of the soundness part is straightforward, the proof of the completeness part consists of the two main constructions:

1. the filtration of the canonical model of L_0^t to prove that L_0^t has the f.m.p.;
2. the construction of p-morphisms from every cone of $\mathbb{I}_{\mathbb{R}}$ to the rooted finite L_0^t-frames.

By the p-morphism lemma the logic of $\mathbb{I}_{\mathbb{R}}$ is the subset of the logic of these frames. Thus it's the subset of L_0^t.

Let us recall that a *p-morphism* $F_1 \twoheadrightarrow F_2$ for $F_1 = (W_1, R_1)$, $F_2 = (W_2, R_2)$ is a surjective function $f : W_1 \to W_2$ such that $\forall a \in W_1\, (f(R_1(a)) = R_2(f(a)))$.

The filtration

Let φ be an L_0^t-consistent formula, let $\mathfrak{M}_c$ be the canonical model of L_0^t and let $\mathfrak{M}$ be the submodel generated by a_φ. The construction of the filtrated model $\mathfrak{M}^+$ consists of three steps: $\mathfrak{M}', \mathfrak{M}^\star$ and $\mathfrak{M}^+$ (with the frames $F', F^\star$ and F^+). For each model we prove the filtration lemma and the following propositions.

Proposition 1 F' validates $A4, AD, AD^{-1}, Ad_2$, Ad_2^{-1}, ACL, ACL^{-1} (all the axioms from L_0^t except $ASBC, ARP$).

Proposition 2 $F^\star$ validates all the axioms from L_0^t except ARP.

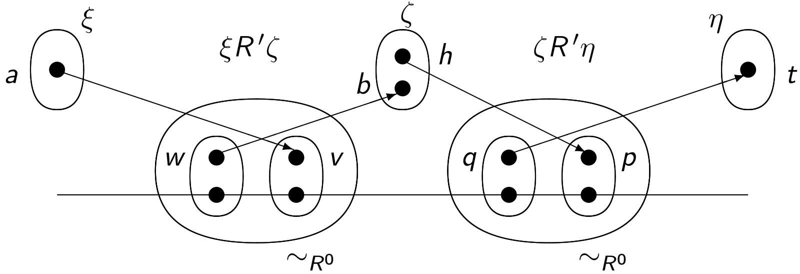
Proposition 3 F^+ is L_0^t-frame.

The filtration, step 1

At the first step we define the filtrated model $\mathfrak{M}'$.
Let $Sub(\varphi)$ be the set of subformulas in φ and let $Cl(\varphi) := Sub(\varphi) \cup \{\neg\psi \mid \psi \in Sub(\varphi)\}$. Let $\sim_{Cl(\varphi)}$ be the equivalence relation on $\underline{\mathfrak{M}}$ defined as follows:
$a \sim_{Cl(\varphi)} b := \forall\psi \in Cl(\varphi)(\psi \in a \Leftrightarrow \psi \in b)$.

Let $\underline{\mathfrak{M}}' := \underline{\mathfrak{M}}/_{\sim_{Cl(\varphi)}}$ and let $[a]$ be the equivalence class for a.

Put $\xi \rightsquigarrow_s \eta$ if for all $\Box\psi$ in $Cl(\varphi)$ $(\Box\psi \in \xi \Rightarrow \Box\psi \in \eta \wedge \psi \in \eta)$ (the relation in the Lemmon filtration),
$\xi R^0 \eta := \xi \rightsquigarrow_s \eta \wedge \exists a \exists b([a]=\xi \wedge aRa \wedge [b]=\eta \wedge bRb)$,
$\xi R' \eta := \exists a \exists b \exists v \exists w([a]=\xi \wedge aRv \wedge [v] \sim_{R^0} [w] \wedge wRb \wedge [b]=\eta)$
where $\xi \sim_{R^0} \eta$ is an abbreviation for $\xi R^0 \eta \wedge \eta R^0 \xi$.

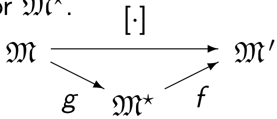


The valuation in $\mathfrak{M}'$ is defined in the standard way.

<u>Lemma 1</u> (the filtration lemma)
$\forall \psi \in Cl(\varphi)\ \forall a \in \mathfrak{M}\ (\mathfrak{M}', [a] \vDash \psi \Leftrightarrow \mathfrak{M}, a \vDash \psi)$.

The filtration, step 2

At the second step we construct the model $\mathfrak{M}^\star$ by adding new singletons to $\mathfrak{M}'$ to eliminate the occurrences of the adjacent clusters containing two or more elements each (we skip the technical details here). Then we construct the surjective functions f, g such that the following diagram is transitive. g is an equivalent of $[\cdot]$ for $\mathfrak{M}^\star$.



The relation $\rightsquigarrow_s$ is defined for $\underline{\mathfrak{M}^\star}$ in the same way as for $\mathfrak{M}'$.
Put $xR^{\circledast}y := x \rightsquigarrow_s y \wedge \exists a \exists b(g(a)=x \wedge aRa \wedge g(b)=y \wedge bRb \wedge ((f(x) \neq x \vee f(y) \neq y) \Rightarrow aRb))$,
$xR^\star y := \exists a \exists b \exists v \exists w(g(a)=x \wedge aRv \wedge g(v) \sim_{R^{\circledast}} g(w) \wedge wRb \wedge g(b)=y)$ where $x \sim_{R^{\circledast}} y$ is an abbreviation for $xR^{\circledast}y \wedge yR^{\circledast}x$.
The valuation in $\mathfrak{M}^\star$ for the variables in $Cl(\varphi)$ is inherited from $\mathfrak{M}$.
<u>Lemma 2</u> (the filtration lemma)
$\forall \psi \in Cl(\varphi)\ \forall a \in \mathfrak{M}\ (\mathfrak{M}^\star, g(a) \vDash \psi \Leftrightarrow \mathfrak{M}, a \vDash \psi)$.

The filtration, step 3

At the third step the model $\mathfrak{M}^+$ is constructed. We prove that non-degenerate clusters in $\mathfrak{M}^\star$ are linearly ordered and choose one reflexive point from the preimage of every non-degenerate cluster. Let $\mathcal{D}$ be the set of all this points.

Proposition The axiom ARP is canonical.

Corollary Let $a, b \in \mathcal{D}$ and $a <_R b$. Then there is $h = h_{a,b}$ in $\mathfrak{M}$ such that $R^{-1}(h) = R^{-1}(a)$ and $R(h) = R(b)$.

Let $\underline{\mathfrak{M}}^+ := \underline{\mathfrak{M}}^\star \cup \{h_{a,b} \,|\, a, b \in \mathcal{D} \text{ and } a <_R b\}$ and let surjective $d : \underline{\mathfrak{M}}^+ \to \underline{\mathfrak{M}}^\star$ be defined as follows:

$$d(p) := \begin{cases} p & \text{if } p \in \underline{\mathfrak{M}}^\star; \\ g(p) & \text{if } p \in \underline{\mathfrak{M}}. \end{cases}$$

The relation $\rightsquigarrow_s$ is defined for $\underline{\mathfrak{M}}^+$ in the same way as for $\mathfrak{M}'$. The relation R^+ is defined as follows:
$pR^{\oplus}q := d(p) = p \wedge d(q) = q \wedge pR^{\circledast}q$.
$pR^+q := \exists v \exists w((d(p) = p \wedge \exists a(g(a) = p \wedge aRv) \vee d(p) \neq p \wedge pRv) \wedge g(v) \sim_{R^{\circledast}} g(w) \wedge (d(q) \neq q \wedge wRq \vee \exists b(g(b) = q \wedge wRb)))$.
The valuation $\nu_{\mathfrak{M}^+}(t)$ for the variables from $Cl(\varphi)$ is
$\nu_{\mathfrak{M}^+}(t) := (\nu_{\mathfrak{M}}(t) \cap \underline{\mathfrak{M}}^+) \cup \nu_{\mathfrak{M}^\star}(t)$ and arbitrary otherwise.
<u>Lemma 3</u> (the filtration lemma)
1. $\forall \psi \in Cl(\varphi)\ \forall a \in \mathfrak{M}\ (\mathfrak{M}^+, g(a) \vDash \psi \Leftrightarrow \mathfrak{M}, a \vDash \psi)$;
2. $\forall \psi \in Cl(\varphi)\ \forall a \in \mathfrak{M} \cap \mathfrak{M}^+\ (\mathfrak{M}^+, a \vDash \psi \Leftrightarrow \mathfrak{M}, a \vDash \psi)$.
<u>Corollary</u> The models $\mathfrak{M}'$, $\mathfrak{M}^\star$ and $\mathfrak{M}^+$ are equivalent modulo $Cl(\varphi)$.
Both the constructions of $\mathfrak{M}^\star$ and $\mathfrak{M}^+$ have similarities with the construction of the finite temporal models in ‘Modal logics with linear alternative relations’, K. Segerberg, 1970.

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